

Categorical representations, KLR algebras and Koszul duality

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Abstract

The parabolic category \mathcal{O} for affine \mathfrak{gl}_N at level $-N - e$ admits a structure of a categorical representation of $\widetilde{\mathfrak{sl}}_e$ with respect to some endofunctors E and F . This category contains a smaller category \mathbf{A} that categorifies the higher level Fock space. We prove that the functors E and F in the category \mathbf{A} are Koszul dual to Zuckerman functors.

The key point of the proof is to show that the functor F for the category \mathbf{A} at level $-N - e$ can be decomposed in terms of components of the functor F for the category \mathbf{A} at level $-N - e - 1$. To prove this, we use the approach of categorical representations. We prove a general fact about categorical representations: a category with an action of $\widetilde{\mathfrak{sl}}_{e+1}$ contains a subcategory with an action of $\widetilde{\mathfrak{sl}}_e$. To prove this claim, we construct an isomorphism between the KLR algebra associated with the quiver $A_{e-1}^{(1)}$ and a subquotient of the KLR algebra associated with the quiver $A_e^{(1)}$.

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1 Introduction

Let O_{-e}^ν be the parabolic category \mathcal{O} with parabolic type ν of the affine version of the Lie algebra \mathfrak{gl}_N at level $-N - e$. In [19], a categorical representation of the affine Kac-Moody algebra $\widetilde{\mathfrak{sl}}_e$ in O_{-e}^ν is considered. Roughly, this means that there are exact biadjoint functors $E_i, F_i: O_{-e}^\nu \rightarrow O_{-e}^\nu$ for $i \in [0, e-1]$ which induce a representation of the Lie algebra $\widetilde{\mathfrak{sl}}_e$ on the Grothendieck group $[O_{-e}^\nu]$ of O_{-e}^ν . The definition of a categorical representation is given in Section 3.4. The category O_{-e}^ν admits a decomposition

$$O_{-e}^\nu = \bigoplus_{\mu \in \mathbb{Z}^e} O_\mu^\nu$$

that lifts the decomposition of the $\widetilde{\mathfrak{sl}}_e$ -module $[O_{-e}^\nu]$ in a direct sum of weight spaces.

The category O_μ^ν is Koszul by [21]. Its Koszul dual category is the category $O_{\nu,+}^\mu$ defined similarly to O_ν^μ at a positive level. In particular, the Koszul duality exchanges the parameter ν (*the parabolic type*) with the parameter μ (*the singular type*). The Koszul duality yields an equivalence of bounded derived categories $D^b(O_\mu^\nu) \simeq D^b(O_{\nu,+}^\mu)$. More details about the Koszul duality can be found in [3].

Let $\alpha_0, \dots, \alpha_{e-1}$ be the simple roots of $\widetilde{\mathfrak{sl}}_e$. We have

$$E_i(O_\mu^\nu) \subset O_{\mu+\alpha_i}^\nu, \quad F_i(O_\mu^\nu) \subset O_{\mu-\alpha_i}^\nu.$$

The aim of this paper is to prove that Koszul dual functors

$$D^b(O_{\nu,+}^\mu) \rightarrow D^b(O_{\nu,+}^{\mu+\alpha_i}), \quad D^b(O_{\nu,+}^\mu) \rightarrow D^b(O_{\nu,+}^{\mu-\alpha_i})$$

to the functors

$$E_i: D^b(O_\mu^\nu) \rightarrow D^b(O_{\mu+\alpha_i}^\nu), \quad F_i: D^b(O_\mu^\nu) \rightarrow D^b(O_{\mu-\alpha_i}^\nu).$$

are the Zuckerman functors.

Unfortunately, we cannot solve this problem for the full category \mathcal{O} . But we are able to do this for a subcategory \mathbf{A} of \mathcal{O} .

By definition, the Zuckerman functor is a composition of a parabolic inclusion functor with a parabolic truncation functor. Thus it is natural to try to decompose the functors E_i and F_i in "smaller" functors. We want to find such a decomposition for F in the following way.

Let \overline{O}_μ^ν , \overline{E}_i , \overline{F}_i be defined in the same way as O_μ^ν , E_i , F_i with e replaced by $e + 1$. Let $\overline{\alpha}_0, \dots, \overline{\alpha}_e$ be the simple roots of $\widetilde{\mathfrak{sl}}_{e+1}$. Fix $k \in [0, e - 1]$. For an e -tuple $\mu = (\mu_1, \dots, \mu_e)$ we set

$$\overline{\mu} = \begin{cases} (\mu_1, \dots, \mu_k, 0, \mu_{k+1}, \dots, \mu_e) & \text{if } k \neq 0, \\ (0, \mu_1, \dots, \mu_e) & \text{if } k = 0. \end{cases}$$

Note that we have $\overline{(\mu - \alpha_k)} = \overline{\mu} - \overline{\alpha}_k - \overline{\alpha}_{k+1}$.

By [8], there is an equivalence of categories $\theta: O_\mu^\nu \rightarrow \overline{O}_{\overline{\mu}}^\nu$. The direct sum of such equivalences identifies the category O_{-e}^ν with a direct factor of the category \overline{O}_{-e-1}^ν . We want to compare the $\widetilde{\mathfrak{sl}}_e$ -action on O_{-e}^ν with the $\widetilde{\mathfrak{sl}}_{e+1}$ -action on \overline{O}_{-e-1}^ν . More precisely, we want to prove the following conjecture.

Conjecture 1.1. *The following diagram of functors is commutative.*

$$\begin{array}{ccccc} \overline{O}_\mu^\nu & \xrightarrow{\overline{F}_k} & \overline{O}_{\overline{\mu} - \overline{\alpha}_k}^\nu & \xrightarrow{\overline{F}_{k+1}} & \overline{O}_{\overline{\mu} - \overline{\alpha}_k - \overline{\alpha}_{k+1}}^\nu \\ \theta \uparrow & & & & \downarrow \theta^{-1} \\ O_\mu^\nu & \xrightarrow{F_k} & O_{\mu - \alpha_k}^\nu & & \end{array} \quad (1)$$

Our motivation is the following. Assume that the conjecture holds. Then we can prove in the same way as in [16] that the functor \overline{F}_k is Koszul dual to the parabolic inclusion functor and the functor \overline{F}_{k+1} is Koszul dual to the parabolic truncation functor. Then we can deduce that F_k is Koszul dual to the Zuckerman functor (which is the composition of the parabolic inclusion functor and the parabolic truncation functor). Thus the problem is reduced to the proof of this conjecture.

It is not hard to see that the diagram from Conjecture 1.1 is commutative at the level of Grothendieck groups. In the case of the category \mathcal{O} of \mathfrak{gl}_N (instead of affine \mathfrak{gl}_N) this is already enough to prove the analogue of Conjecture 1.1, using the theory of projective functors. Indeed, [1, Thm. 3.4] implies that two projective functors are isomorphic if their actions on the Grothendieck group coincide. Unfortunately, there is no satisfactory theory of projective functors for the affine case (an attempt to develop such a theory was given in [9]).

We choose another strategy to prove this conjecture. The idea is to relate the notion of a categorical representation of $\widetilde{\mathfrak{sl}}_e$ with the notion of a categorical representation of $\widetilde{\mathfrak{sl}}_{e+1}$. This can be done using the following inclusion of Lie

algebras $\widetilde{\mathfrak{sl}}_e \subset \widetilde{\mathfrak{sl}}_{e+1}$

$$e_r \mapsto \begin{cases} e_r & \text{if } r \in [0, k-1], \\ [e_k, e_{k+1}] & \text{if } r = k, \\ e_{r+1} & \text{if } r \in [k+1, e-1], \end{cases}$$

$$f_r \mapsto \begin{cases} f_r & \text{if } r \in [0, k-1], \\ [f_{k+1}, f_k] & \text{if } r = k, \\ f_{r+1} & \text{if } r \in [k+1, e-1]. \end{cases}$$

First, we recall the notion of a categorical representation. Let \mathbf{k} be a field. Let \mathcal{C} be an abelian Hom-finite \mathbf{k} -linear category that admits a direct sum decomposition $\mathcal{C} = \bigoplus_{\mu \in \mathbb{Z}^e} \mathcal{C}_\mu$. A categorical representation of $\widetilde{\mathfrak{sl}}_e$ in \mathcal{C} is a pair of biadjoint functors $\overline{E}_i, F_i: \mathcal{C} \rightarrow \mathcal{C}$ for $i \in [0, e-1]$ satisfying a list of axioms. The main axiom is that for each $d \in \mathbb{N}$ there is an algebra homomorphism $R_d \rightarrow \text{End}(F^d)^{\text{op}}$, where R_d is the KLR algebra of rank d associated with the quiver $A_{e-1}^{(1)}$.

Now we explain our main result about categorical representations. Let $\overline{\mathcal{C}}$ be an abelian Hom-finite \mathbf{k} -linear category. Assume that $\overline{\mathcal{C}} = \bigoplus_{\mu \in \mathbb{Z}^{e+1}} \overline{\mathcal{C}}_\mu$ has a structure of a categorical representation of $\widetilde{\mathfrak{sl}}_{e+1}$ with respect to functors $\overline{E}_i, \overline{F}_i$ for $i \in [0, e]$. Assume additionally that the subcategory $\overline{\mathcal{C}}_\mu$ is zero whenever μ has a negative entry. For each e -tuple $\mu \in \mathbb{N}^e$ we consider the $(e+1)$ -tuple $\overline{\mu}$ as above and we set $\mathcal{C}_\mu = \overline{\mathcal{C}}_{\overline{\mu}}$,

$$\mathcal{C} = \bigoplus_{\mu \in \mathbb{N}^e} \mathcal{C}_\mu.$$

Next, consider the endofunctors of \mathcal{C} given by

$$E_i = \begin{cases} \overline{E}_i|_{\mathcal{C}} & \text{if } 0 \leq i < k, \\ \overline{E}_k \overline{E}_{k+1}|_{\mathcal{C}} & \text{if } i = k, \\ \overline{E}_{i+1}|_{\mathcal{C}} & \text{if } k < i < e, \end{cases}$$

$$F_i = \begin{cases} \overline{F}_i|_{\mathcal{C}} & \text{if } 0 \leq i < k, \\ \overline{F}_{k+1} \overline{F}_k|_{\mathcal{C}} & \text{if } i = k, \\ \overline{F}_{i+1}|_{\mathcal{C}} & \text{if } k < i < e. \end{cases}$$

The following theorem holds.

Theorem 1.2. *The category \mathcal{C} has a structure of a categorical representation of $\widetilde{\mathfrak{sl}}_e$ with respect to the functors $E_0, \dots, E_{e-1}, F_0, \dots, F_{e-1}$.*

The proof of Theorem 1.2 can be reduced to comparison the KLR algebras associated with the quivers of types $A_{e-1}^{(1)}, A_e^{(1)}$, denoted by $\Gamma, \overline{\Gamma}$ respectively. This is done in Section 2.

Let $\alpha = \sum_{i=0}^{e-1} d_i \alpha_i$ be a dimension vector of the quiver Γ . We consider the dimension vector $\bar{\alpha}$ of $\bar{\Gamma}$ defined in the following way

$$\bar{\alpha} = \sum_{i=0}^k d_i \alpha_i + \sum_{i=k+1}^e d_{i-1} \alpha_i.$$

Let $R_\alpha(\Gamma)$ (resp. $R_{\bar{\alpha}}(\bar{\Gamma})$) be the KLR algebra associated with the quiver Γ and the dimension vector α (resp. the quiver $\bar{\Gamma}$ and the dimension vector $\bar{\alpha}$). The algebra $R_{\bar{\alpha}}(\bar{\Gamma})$ contains idempotents $e(\mathbf{i})$ parameterized by sequences \mathbf{i} of vertices of $\bar{\Gamma}$. In Section 2.4 we consider some sets of such sequences $\bar{T}_{\text{ord}}^{\bar{\alpha}}$ and $\bar{T}_{\text{un}}^{\bar{\alpha}}$. Set $\mathbf{e} = \sum_{\mathbf{i} \in \bar{T}_{\text{ord}}^{\bar{\alpha}}} e(\mathbf{i}) \in R_{\bar{\alpha}}(\bar{\Gamma})$ and

$$S_{\bar{\alpha}}(\bar{\Gamma}) = \mathbf{e} R_{\bar{\alpha}}(\bar{\Gamma}) \mathbf{e} / \sum_{\mathbf{i} \in \bar{T}_{\text{un}}^{\bar{\alpha}}} \mathbf{e} R_{\bar{\alpha}}(\bar{\Gamma}) e(\mathbf{i}) R_{\bar{\alpha}}(\bar{\Gamma}) \mathbf{e}.$$

We call $S_{\bar{\alpha}}(\bar{\Gamma})$ a *balanced KLR algebra*. The main result of Section 2 is the following theorem.

Theorem 1.3. *There is an algebra isomorphism $R_\alpha(\Gamma) \simeq S_{\bar{\alpha}}(\bar{\Gamma})$. \square*

Now we explain why Theorem 1.2 can be useful to prove Conjecture 1.1. Take $\bar{\mathcal{C}} = \bar{\mathcal{O}}_{-(e+1)}^\nu$. Then the subcategory $\mathcal{C} \subset \bar{\mathcal{C}}$ as in Theorem 1.2 is equivalent to \mathcal{O}_{-e}^ν . We get two categorical representations of \mathfrak{sl}_e in \mathcal{O}_{-e}^ν :

- the original one,
- the \mathfrak{sl}_e -categorical representation structure induced from the \mathfrak{sl}_{e+1} -categorical representation structure in $\bar{\mathcal{O}}_{-(e+1)}^\nu$.

It is enough to prove that these two categorical representation structures are the same to prove Conjecture 1.1.

Unfortunately we cannot apply the uniqueness theorem for categorical representations because the \mathfrak{sl}_e -module categorified by \mathcal{O}_{-e}^ν is not simple. However, we can obtain a weaker version of Conjecture 1.1. The category \mathcal{O}_{-e}^ν contains subcategories $\mathbf{A}^\nu[\alpha]$, parameterized by dimension vectors α of Γ . The direct sum of such categories categorifies the Fock space. In this case we can use the technique similar to one used in [19]. This technique allows to prove in some cases that two categorical representations that categorify the Fock space are the same. We get the following.

For $i \in [0, e-1]$ we have $F_i(\mathbf{A}^\nu[\alpha]) \subset \mathbf{A}^\nu[\alpha + \alpha_i]$. Let $\bar{\mathbf{A}}^\nu[\bar{\alpha}]$ be defined in the same way as $\mathbf{A}^\nu[\alpha]$ with respect to the parameter $e+1$ instead of e . Let $|\alpha|$ be the height of α (i.e., we have $|\alpha| = \sum_i d_i$). The main result of Section 3 is the following theorem.

Theorem 1.4. *Assume that $e > 2$ and $\nu = (\nu_1, \dots, \nu_l)$ satisfies $\nu_r > |\alpha|$ for each $r \in [1, l]$. There exists a dimension vector β for $\bar{\Gamma}$ such that for each dimension vector α for Γ there are equivalences of categories $\theta'_\alpha: \mathbf{A}^\nu[\alpha] \rightarrow \bar{\mathbf{A}}^\nu[\beta + \bar{\alpha}]$*

and $\theta'_{\alpha+\alpha_k} : \mathbf{A}^\nu[\alpha+\alpha_k] \rightarrow \overline{\mathbf{A}}^\nu[\beta+\overline{\alpha}+\overline{\alpha}_k+\overline{\alpha}_{k+1}]$ such that the following diagram is commutative

$$\begin{array}{ccc} \overline{\mathbf{A}}^\nu[\beta+\overline{\alpha}] & \xrightarrow{\overline{F}_{k+1}\overline{F}_k} & \overline{\mathbf{A}}^\nu[\beta+\overline{\alpha}+\overline{\alpha}_k+\overline{\alpha}_{k+1}] \\ \theta'_\alpha \uparrow & & \theta'_{\alpha+\alpha_k} \uparrow \\ \mathbf{A}^\nu[\alpha] & \xrightarrow{F_k} & \mathbf{A}^\nu[\beta+\alpha+\alpha_k]. \end{array}$$

□

The technique of [19] uses essentially the deformation argument. To make it applicable in our situation we have to find a version of Theorem 1.3 over a local ring. This is done in Lemma 2.21.

The paper has the following structure. In Section 2 we study KLR algebras. In particular, we prove Theorem 1.3 and its deformed version over a local ring. In Section 3 we study categorical representations. We prove our main result about categorical representations (Theorem 1.2). Next, we use the categorical representations to decompose the functor F in the category \mathbf{A} (Theorem 1.4). In Section 4 we prove that in some cases the functors E and F for the category \mathcal{O} admit graded lifts. In Section 5 we prove that the functors E and F for the category \mathbf{A} are Koszul dual to Zuckerman functors. We deduce this from the main results of Sections 3, 4 using an approach similar to [16]. In Appendix A we generalize Lemma 3.9 to arbitrary symmetric Kac-Moody Lie algebras. In Appendix B we give a geometric construction of the isomorphism Φ in Theorem 2.11.

2 KLR algebras and Hecke algebras

For a noetherian ring A we denote by $\text{mod}(A)$ the abelian category of left finitely generated A -modules. We denote by \mathbb{N} the set of non-negative integers. By a commutative diagram of functors we always mean a diagram that commutes up to an isomorphism of functors.

2.1 Kac-Moody algebras associated with a quiver

Let $\Gamma = (I, H)$ be a quiver without 1-loops with the set of vertices I and the set of arrows H . For $i, j \in I$ let $h_{i,j}$ be the number of arrows from i to j and set also $a_{i,j} = 2\delta_{i,j} - h_{i,j} - h_{j,i}$. Let \mathfrak{g}_I be the Kac-Moody algebra over \mathbb{C} associated with the matrix $(a_{i,j})$. Denote by e_i, f_i for $i \in I$ the Serre generators of \mathfrak{g}_I .

For each $i \in I$ let $\alpha_i, \check{\alpha}_i$ be the simple root and coroot corresponding to e_i and let Λ_i be the fundamental weight. Set

$$Q_I = \bigoplus_{i \in I} \mathbb{Z}\alpha_i, \quad Q_I^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i, \quad P_I = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i, \quad P_I^+ = \bigoplus_{i \in I} \mathbb{N}\Lambda_i.$$

Let X_I be the free abelian group with basis $\{\varepsilon_i; i \in I\}$. Set also

$$X_I^+ = \bigoplus_{i \in I} \mathbb{N}\varepsilon_i. \quad (2)$$

For $\alpha \in Q_I^+$ denote by $|\alpha|$ its height. Set $I^\alpha = \{\mathbf{i} = (i_1, \dots, i_{|\alpha|}) \in I^{|\alpha|}; \sum_{r=1}^{|\alpha|} \alpha_{i_r} = \alpha\}$.

Let $\Gamma_\infty = (I_\infty, H_\infty)$ be the quiver with the set of vertices $I_\infty = \mathbb{Z}$ and the set of arrows $H_\infty = \{i \rightarrow i+1; i \in I_\infty\}$. Assume that $e > 1$ is an integer. Let $\Gamma_e = (I_e, H_e)$ be the quiver with the set of vertices $I_e = \mathbb{Z}/e\mathbb{Z}$ and the set of arrows $H_e = \{i \rightarrow i+1; i \in I_e\}$. Then \mathfrak{g}_{I_e} is the Lie algebra $\tilde{\mathfrak{sl}}_e = \mathfrak{sl}_e \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$.

Assume that $\Gamma = (I, H)$ is a quiver whose connected components are of the form Γ_e , with $e \in \mathbb{N}$, $e > 1$ or $e = \infty$. For $i \in I$ denote by $i+1$ and $i-1$ the (unique) vertices in I such that there are arrows $i \rightarrow i+1$, $i-1 \rightarrow i$. Let us also consider the following additive map

$$\iota: Q_I \rightarrow X_I, \quad \alpha_i \mapsto \varepsilon_i - \varepsilon_{i+1}.$$

Fix a formal variable δ and set $X_I^\delta = X_I \oplus \mathbb{Z}\delta$. We can lift the \mathbb{Z} -linear map ι to a \mathbb{Z} -linear map

$$\iota^\delta: Q_I \rightarrow X_I^\delta, \quad \alpha_i \mapsto \varepsilon_i - \varepsilon_{i+1} - \delta.$$

Note that the map $\iota^\delta: Q_I \rightarrow X_I^\delta$ is injective (while ι is not injective). We may omit the symbols ι , ι^δ and write α instead of $\iota(\alpha)$ or $\iota^\delta(\alpha)$.

We will sometimes abbreviate

$$Q_e = Q_{I_e}, \quad X_e = X_{I_e}, \quad X_e^\delta = X_{I_e}^\delta, \quad P_e = P_{I_e}.$$

2.2 Doubled quiver

Let $\Gamma = (I, H)$ be a quiver without 1-loops. Fix a decomposition $I = I_0 \sqcup I_1$ such that there are no arrows between the vertices in I_1 . In this section we define a *doubled quiver* $\bar{\Gamma} = (\bar{I}, \bar{H})$ associated with (Γ, I_0, I_1) . The idea is to "double" each vertex in the set I_1 (we do not touch the vertices from I_0). We replace each vertex $i \in I_1$ by a couple of vertices i^1 and i^2 with an arrow $i^1 \rightarrow i^2$. Each arrow entering to i should be replaced by an arrow entering to i^1 , each arrow coming from i should be replaced by an arrow coming from i^2 .

Now we describe the construction of $\bar{\Gamma} = (\bar{I}, \bar{H})$ formally. Let \bar{I}_0 be a set that is in bijection with I_0 . Let i^0 be the element of \bar{I}_0 associated with an element $i \in I_0$. Similarly, let \bar{I}_1 and \bar{I}_2 be sets that are in bijection with I_1 . Denote by i^1 and i^2 the element of \bar{I}_1 and \bar{I}_2 respectively that correspond to an element $i \in I_1$. Put $\bar{I} = \bar{I}_0 \sqcup \bar{I}_1 \sqcup \bar{I}_2$. We define \bar{H} in the following way. The set \bar{H} contains 4 types of arrows:

- an arrow $i^0 \rightarrow j^0$ for each arrow $i \rightarrow j$ in H with $i, j \in I_0$,

- an arrow $i^0 \rightarrow j^1$ for each arrow $i \rightarrow j$ in H with $i \in I_0, j \in I_1$,
- an arrow $i^2 \rightarrow j^0$ for each arrow $i \rightarrow j$ in H with $i \in I_1, j \in I_0$,
- an arrow $i^1 \rightarrow i^2$ for each vertex $i \in I_1$.

Set $I^\infty = \coprod_{d \in \mathbb{N}} I^d$, $\bar{I}^\infty = \coprod_{d \in \mathbb{N}} \bar{I}^d$, where I^d, \bar{I}^d are the cartesian products. The concatenation yields a monoid structure on I^∞ and \bar{I}^∞ . Let $\phi: I^\infty \rightarrow \bar{I}^\infty$ be the unique morphism of monoids such that for $i \in I \subset I^\infty$ we have

$$\phi(i) = \begin{cases} i^0 & \text{if } i \in I_0, \\ (i^1, i^2) & \text{if } i \in I_1. \end{cases}$$

There is a unique \mathbb{Z} -linear map $\phi: Q_I \rightarrow Q_{\bar{I}}$ such that $\phi(I^\alpha) \subset I^{\phi(\alpha)}$ for each $\alpha \in Q_I^+$. It is given by

$$\phi(\alpha_i) = \begin{cases} \alpha_{i^0} & \text{if } i \in I_0, \\ \alpha_{i^1} + \alpha_{i^2} & \text{if } i \in I_1. \end{cases} \quad (3)$$

Let ϕ denote also the unique additive embedding

$$\phi: X_I \rightarrow X_{\bar{I}}, \quad \varepsilon_i \mapsto \varepsilon_{i'}, \quad (4)$$

where

$$i' = \begin{cases} i^0 & \text{if } i \in I_0, \\ i^1 & \text{if } i \in I_1. \end{cases} \quad (5)$$

2.3 KLR algebras

Let \mathbf{k} be a field. Let $\Gamma = (I, H)$ be a quiver without 1-loops. For $r \in [1, d-1]$ let s_r be the transposition $(r, r+1) \in \mathfrak{S}_d$. For $\mathbf{i} = (i_1, \dots, i_d) \in I^d$ set $s_r(\mathbf{i}) = (i_1, \dots, i_{r-1}, i_{r+1}, i_r, i_{r+2}, \dots, i_d)$. For $i, j \in I$ we set

$$Q_{i,j}(u, v) = \begin{cases} 0 & \text{if } i = j, \\ (v - u)^{h_{i,j}} (u - v)^{h_{j,i}} & \text{else.} \end{cases}$$

Definition 2.1. The *KLR-algebra* $R_{d,\mathbf{k}}(\Gamma)$ is the \mathbf{k} -algebra with the set of generators $\tau_1, \dots, \tau_{d-1}, x_1, \dots, x_d, e(\mathbf{i})$ where $\mathbf{i} \in I^d$, modulo the following defining relations

- $e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}e(\mathbf{i})$,
- $\sum_{\mathbf{i} \in I^d} e(\mathbf{i}) = 1$,
- $x_r e(\mathbf{i}) = e(\mathbf{i}) x_r$,
- $\tau_r e(\mathbf{i}) = e(s_r(\mathbf{i})) \tau_r$,
- $x_r x_s = x_s x_r$,

- $\tau_r x_{r+1} e(\mathbf{i}) = (x_r \tau_r + \delta_{i_r, i_{r+1}}) e(\mathbf{i}),$
- $x_{r+1} \tau_r e(\mathbf{i}) = (\tau_r x_r + \delta_{i_r, i_{r+1}}) e(\mathbf{i}),$
- $\tau_r x_s = x_s \tau_r, \text{ if } s \neq r, r+1,$
- $\tau_r \tau_s = \tau_s \tau_r, \text{ if } |r-s| > 1,$
- $\tau_r^2 e(\mathbf{i}) = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ Q_{i_r, i_{r+1}}(x_r, x_{r+1}) e(\mathbf{i}) & \text{else,} \end{cases}$
- $(\tau_r \tau_{r+1} \tau_r - \tau_{r+1} \tau_r \tau_{r+1}) e(\mathbf{i}) = \begin{cases} (x_{r+2} - x_r)^{-1} (Q_{i_r, i_{r+1}}(x_{r+2}, x_{r+1}) - Q_{i_r, i_{r+1}}(x_r, x_{r+1})) e(\mathbf{i}) & \text{if } i_r = i_{r+2}, \\ 0 & \text{else,} \end{cases}$

for each $\mathbf{i}, \mathbf{j}, r$ and s . We may write $R_{d, \mathbf{k}} = R_{d, \mathbf{k}}(\Gamma)$. The algebra $R_{d, \mathbf{k}}$ admits a \mathbb{Z} -grading such that $\deg e(\mathbf{i}) = 0$, $\deg x_r = 2$, $\deg \tau_s e(\mathbf{i}) = -a_{i_s, i_{s+1}}$, for each $1 \leq r \leq d$, $1 \leq s < d$ and $\mathbf{i} \in I^d$.

For each $\alpha \in Q_I^+$ such that $|\alpha| = d$ set $e(\alpha) = \sum_{\mathbf{i} \in I^\alpha} e(\mathbf{i}) \in R_{d, \mathbf{k}}$. It is a homogeneous central idempotent of degree zero. We have the following decomposition into a sum of unitary \mathbf{k} -algebras $R_{d, \mathbf{k}} = \bigoplus_{|\alpha|=d} R_{\alpha, \mathbf{k}}$, where $R_{\alpha, \mathbf{k}} = e(\alpha) R_{d, \mathbf{k}}$.

Let $\mathbf{k}_d^{(I)}$ be the direct sum of copies of the ring $\mathbf{k}_d[x] := \mathbf{k}[x_1, \dots, x_d]$ labelled by I^d . We write

$$\mathbf{k}_d^{(I)} = \bigoplus_{\mathbf{i} \in I^d} \mathbf{k}_d[x] e(\mathbf{i}), \quad (6)$$

where $e(\mathbf{i})$ is the idempotent of the ring $\mathbf{k}_d^{(I)}$ projecting to the component \mathbf{i} . A polynomial in $\mathbf{k}_d[x]$ can be considered as an element of $\mathbf{k}_d^{(I)}$ via the diagonal inclusion. For each $i, j \in I$ fix a polynomial $P_{i,j}(u, v)$ such that we have $Q_{i,j}(u, v) = P_{i,j}(u, v) P_{j,i}(v, u)$. Then by [17, Sec. 3.2] the algebra $R_{d, \mathbf{k}}$ has a representation in the vector space $\mathbf{k}_d^{(I)}$ such that the element $e(\mathbf{i})$ acts by the projection to $\mathbf{k}_d^{(I)} e(\mathbf{i})$, the element x_r acts by multiplication by x_r and such that for $f \in \mathbf{k}_d[x]$ we have

$$\tau_r \cdot f e(\mathbf{i}) = \begin{cases} (x_r - x_{r+1})^{-1} (s_r(f) - f) e(\mathbf{i}) & \text{if } i_r = i_{r+1}, \\ P_{i_r, i_{r+1}}(x_{r+1}, x_r) s_r(f) e(s_r(\mathbf{i})) & \text{otherwise.} \end{cases} \quad (7)$$

We will always choose $P_{i,j}$ in the following way:

$$P_{i,j}(u, v) = (u - v)^{h_{j,i}}.$$

Remark 2.2. There is an explicit construction of a basis of a KLR algebra. Assume $\mathbf{i}, \mathbf{j} \in I^\alpha$. Set $\mathfrak{S}_{\mathbf{i}, \mathbf{j}} = \{w \in \mathfrak{S}_d; w(\mathbf{i}) = \mathbf{j}\}$. For each permutation $w \in \mathfrak{S}_{\mathbf{i}, \mathbf{j}}$ fix a reduced expression $w = s_{p_1} \cdots s_{p_r}$ and set $\tau_w = \tau_{p_1} \cdots \tau_{p_r}$. Then the vector space $e(\mathbf{j}) R_{\alpha, \mathbf{k}} e(\mathbf{i})$ has a basis $\{\tau_w x_1^{a_1} \cdots x_d^{a_d} e(\mathbf{i}); w \in \mathfrak{S}_{\mathbf{i}, \mathbf{j}}, a_1, \dots, a_d \in \mathbb{N}\}$. Note that the element τ_w depends on the reduced expression of w . Moreover,

if we change the reduced expression of w , then the element $\tau_w e(\mathbf{i})$ changes only by a linear combination of monomials of the form $\tau_{q_1} \cdots \tau_{q_t} x_1^{b_1} \cdots x_d^{b_d} e(\mathbf{i})$ with $t < \ell(w)$.

Fix a weight $\Lambda = \sum_{i \in I} n_i \Lambda_i \in P_I^+$.

Definition 2.3. The *cyclotomic KLR-algebra* $R_{\alpha, \mathbf{k}}^\Lambda$ is the quotient of $R_{\alpha, \mathbf{k}}$ by the two-sided ideal generated by $x_1^{n_{i_1}} e(\mathbf{i})$ for $\mathbf{i} = (i_1, \dots, i_d) \in I^\alpha$.

For each $i \in I$ there is an inclusion $R_{\alpha, \mathbf{k}} \subset R_{\alpha + \alpha_i, \mathbf{k}}$ that takes $e(\mathbf{i})$ to $e(\mathbf{i}, i)$, x_r to $x_r e(\mathbf{i}, i)$, τ_r to $\tau_r e(\mathbf{i}, i)$. It factors through a homomorphism $R_{\alpha, \mathbf{k}}^\Lambda \rightarrow R_{\alpha + \alpha_i, \mathbf{k}}^\Lambda$. Let

$$F_i^\Lambda: \text{mod}(R_{\alpha, \mathbf{k}}^\Lambda) \rightarrow \text{mod}(R_{\alpha + \alpha_i, \mathbf{k}}^\Lambda), \quad E_i^\Lambda: \text{mod}(R_{\alpha + \alpha_i, \mathbf{k}}^\Lambda) \rightarrow \text{mod}(R_{\alpha, \mathbf{k}}^\Lambda)$$

be the induction and restriction functors with respect to this algebra homomorphism.

2.4 Balanced KLR algebras

Fix a decomposition $I = I_0 \sqcup I_1$ as in Section 2.2 and consider the quiver $\bar{\Gamma} = (\bar{I}, \bar{H})$ as in Section 2.2. Recall the decomposition $\bar{I} = \bar{I}_0 \sqcup \bar{I}_1 \sqcup \bar{I}_2$. In this section we work with the KLR algebra associated with the quiver $\bar{\Gamma}$.

We say that a sequence $\mathbf{i} = (i_1, i_2, \dots, i_d) \in \bar{I}^d$ is *unordered* if there is an index $r \in [1, d]$ such that the number of elements from \bar{I}_2 in the sequence (i_1, i_2, \dots, i_r) is strictly greater than the number of elements from \bar{I}_1 . We say that it is *well-ordered* if for each index a such that $i_a = i^1$ for some $i \in I_1$, we have $a < d$ and $i_{a+1} = i^2$. We denote by $\bar{I}_{\text{ord}}^\alpha$ and $\bar{I}_{\text{un}}^\alpha$ the subsets of well-ordered and unordered sequences in \bar{I}^α respectively.

Definition 2.4. For $\bar{\alpha} \in Q_{\bar{I}}^+$, the *balanced quotient* $\tilde{S}_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma})$ of $R_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma})$ is the algebra

$$R_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma}) / \sum_{\mathbf{i} \in \bar{I}_{\text{un}}^\alpha} R_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma}) e(\mathbf{i}) R_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma}).$$

The map ϕ from Section 2.2 yields a bijection

$$\phi: Q_I^+ \rightarrow \{\alpha = \sum_{i \in \bar{I}} d_i \alpha_i \in Q_{\bar{I}}^+; d_{i^1} = d_{i^2}, \forall i \in I_1\}, \quad \alpha \mapsto \bar{\alpha}.$$

Fix $\alpha \in Q_I^+$. Set $\mathbf{e} = \sum_{\mathbf{i} \in \bar{I}_{\text{ord}}^\alpha} e(\mathbf{i}) \in R_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma})$. We also denote by \mathbf{e} the image of \mathbf{e} in $\tilde{S}_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma})$.

Definition 2.5. For $\alpha \in Q_I^+$, the *balanced KLR algebra* is the algebra

$$S_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma}) = \mathbf{e} R_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma}) \mathbf{e} / \sum_{\mathbf{i} \in \bar{I}_{\text{un}}^\alpha} \mathbf{e} R_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma}) e(\mathbf{i}) R_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma}) \mathbf{e}.$$

In other words we have $S_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma}) = \mathbf{e} \tilde{S}_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma}) \mathbf{e}$. We may write $S_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma}) = S_{\bar{\alpha}, \mathbf{k}}$.

Remark 2.6. Assume that $\mathbf{i} = (i_1, \dots, i_d) \in \bar{I}_{\text{ord}}^{\bar{\alpha}}$. Let a be an index such that $i_a \in \bar{I}_1$. We have the relation $\tau_a^2 e(\mathbf{i}) = (x_{a+1} - x_a) e(\mathbf{i})$ in $R_{\bar{\alpha}, \mathbf{k}}$. Moreover, we have $\tau_a^2 e(\mathbf{i}) = \tau_a e(s_a(\mathbf{i})) \tau_a e(\mathbf{i})$ and $s_a(\mathbf{i})$ is unordered. Thus we have $x_a e(\mathbf{i}) = x_{a+1} e(\mathbf{i})$ in $S_{\bar{\alpha}, \mathbf{k}}$.

2.5 The polynomial representation of $S_{\bar{\alpha}, \mathbf{k}}$

We assume $\alpha = \sum_{i \in \bar{I}} d_i \alpha_i \in Q_I^+$. Let $\mathbf{i} = (i_1, \dots, i_d) \in \bar{I}_{\text{ord}}^{\bar{\alpha}}$. Denote by $J(\mathbf{i})$ the ideal of the polynomial ring $\mathbf{k}_d[x]e(\mathbf{i}) \subset \mathbf{k}_d^{(\bar{I})}$ generated by the set

$$\{(x_r - x_{r+1})e(\mathbf{i}); i_r \in \bar{I}_1\}.$$

Lemma 2.7. *Assume that $\mathbf{i} \in \bar{I}_{\text{ord}}^{\bar{\alpha}}$ and $\mathbf{j} \in \bar{I}_{\text{un}}^{\bar{\alpha}}$. Then each element of $e(\mathbf{i})R_{\bar{\alpha}, \mathbf{k}}e(\mathbf{j})$ maps $\mathbf{k}_d[x]e(\mathbf{j})$ to $J(\mathbf{i})$.*

Proof. We will prove by induction that each monomial $\tau_{p_1} \cdots \tau_{p_k}$ such that the permutation $w = s_{p_1} \cdots s_{p_k} \in \mathfrak{S}_d$ satisfies $w(\mathbf{j}) = \mathbf{i}$ maps $\mathbf{k}_d[x]e(\mathbf{j})$ to $J(\mathbf{i})$.

Assume $k = 1$. Write $p = p_1$. Let us write $\mathbf{i} = (i_1, \dots, i_d)$ and $\mathbf{j} = (j_1, \dots, j_d)$. Then we have $\mathbf{i} = s_p(\mathbf{j})$. By assumptions we know that there exists $i \in I_1$ such that $i_p = j_{p+1} = i^1$, $i_{p+1} = j_p = i^2$. In this case the statement is obvious because τ_p maps $f e(\mathbf{j}) \in \mathbf{k}_d[x]e(\mathbf{j})$ to $(x_{p+1} - x_p) s_p(f) e(\mathbf{i})$ by (7).

Now consider a monomial $\tau_{p_1} \cdots \tau_{p_k}$ such that the permutation $w = s_{p_1} \cdots s_{p_k}$ satisfies $w(\mathbf{j}) = \mathbf{i}$ and assume that the statement is true for all such monomials of smaller length. By assumptions on \mathbf{i} and \mathbf{j} there is an index $r \in [1, d]$ such that $i_r = i^1$ for some $i \in I_1$ and $w^{-1}(r+1) < w^{-1}(r)$. Thus w has a reduced expression of the form $w = s_r s_{r_1} \cdots s_{r_h}$. This implies that $\tau_{p_1} \cdots \tau_{p_k} e(\mathbf{j})$ is equal to a monomial of the form $\tau_r \tau_{r_1} \cdots \tau_{r_h} e(\mathbf{j})$ modulo monomials of the form $\tau_{q_1} \cdots \tau_{q_t} x_1^{b_1} \cdots x_d^{b_d} e(\mathbf{j})$ with $t < k$, see Remark 2.2. As the sequence $s_r(\mathbf{i})$ is unordered, the case $k = 1$ and the induction hypothesis imply the statement. \square

Lemma 2.8. *Assume that $\mathbf{i}, \mathbf{j} \in \bar{I}_{\text{ord}}^{\bar{\alpha}}$. Then each element of $e(\mathbf{i})R_{\bar{\alpha}, \mathbf{k}}e(\mathbf{j})$ maps $J(\mathbf{j})$ into $J(\mathbf{i})$.*

Proof. Take $y \in e(\mathbf{i})R_{\bar{\alpha}, \mathbf{k}}e(\mathbf{j})$. We must prove that for each $r \in [1, d]$ such that $j_r = i^1$ for some $i \in I_1$ and each $f \in \mathbf{k}_d[x]$ we have $y((x_r - x_{r+1})f e(\mathbf{j})) \in J(\mathbf{i})$. We have $(x_r - x_{r+1})f e(\mathbf{j}) = -\tau_r^2(f e(\mathbf{i}))$. This implies

$$y((x_r - x_{r+1})f e(\mathbf{j})) = -y\tau_r^2(f e(\mathbf{i})) = -y\tau_r e(s_r(\mathbf{j}))(\tau_r(f e(\mathbf{j}))).$$

Thus Lemma 2.7 implies the statement because the sequence $s_r(\mathbf{j})$ is unordered. \square

The representation of $R_{\bar{\alpha}, \mathbf{k}}$ on

$$\mathbf{k}_{\bar{\alpha}}^{(\bar{I})} := \bigoplus_{\mathbf{i} \in \bar{I}^{\bar{\alpha}}} \mathbf{k}_{|\bar{\alpha}|}[x]e(\mathbf{i})$$

yields a representation of $\mathbf{e}R_{\bar{\alpha}, \mathbf{k}}\mathbf{e}$ on

$$\mathbf{k}_{\bar{\alpha}, \text{ord}}^{(\bar{I})} := \bigoplus_{\mathbf{i} \in \bar{I}_{\text{ord}}^{\bar{\alpha}}} \mathbf{k}_{|\bar{\alpha}|}[x]e(\mathbf{i}).$$

Set $J_{\bar{\alpha}, \text{ord}} = \bigoplus_{\mathbf{i} \in \bar{I}_{\text{ord}}^{\bar{\alpha}}} J(\mathbf{i})$. From Lemmas 2.7, 2.8 we deduce the following.

Lemma 2.9. *The representation of $R_{\bar{\alpha}, \mathbf{k}}$ on $\mathbf{k}_{\bar{\alpha}}^{(\bar{I})}$ factors through a representation of $S_{\bar{\alpha}, \mathbf{k}}$ on $\mathbf{k}_{\bar{\alpha}, \text{ord}}^{(\bar{I})}/J_{\bar{\alpha}, \text{ord}}$. This representation is faithful.*

Proof. The faithfulness is proved in the proof of Theorem 2.11. \square

2.6 The comparison of the polynomial representations

Fix $\alpha \in Q_I^+$. Set $d = |\alpha|$ and $\bar{d} = |\bar{\alpha}|$. For each sequence $\mathbf{i} = (i_1, \dots, i_d) \in I^\alpha$ and $r \in [1, d]$ we denote by r' or r'_i the positive integer such that $r' - 1$ is the length of the sequence $\phi(i_1, \dots, i_{r-1}) \in \bar{I}^\infty$.

For $r \in [1, d]$ (resp. $r \in [1, d-1]$) consider the element $x_r^* \in S_{\bar{\alpha}, \mathbf{k}}$ (resp. $\tau_r^* \in S_{\bar{\alpha}, \mathbf{k}}$) such that for each $\mathbf{i} \in I^\alpha$ we have

$$x_r^* e(\phi(\mathbf{i})) = x_{r'} e(\phi(\mathbf{i})),$$

$$\tau_r^* e(\phi(\mathbf{i})) = \begin{cases} \tau_{r'} e(\phi(\mathbf{i})), & \text{if } i_r, i_{r+1} \in I_0, \\ \tau_{r'} \tau_{r'+1} e(\phi(\mathbf{i})), & \text{if } i_r \in I_1, i_{r+1} \in I_0, \\ \tau_{r'+1} \tau_{r'} e(\phi(\mathbf{i})), & \text{if } i_r \in I_0, i_{r+1} \in I_1, \\ \tau_{r'+1} \tau_{r'+2} \tau_{r'} \tau_{r'+1} e(\phi(\mathbf{i})), & \text{if } i_r, i_{r+1} \in I_1, i_r \neq i_{r+1}, \\ -\tau_{r'+1} \tau_{r'+2} \tau_{r'} \tau_{r'+1} e(\phi(\mathbf{i})), & \text{if } i_r = i_{r+1} \in I_1. \end{cases}$$

For each $\mathbf{i} \in I^\alpha$ we have the algebra isomorphism

$$\mathbf{k}_d[x]e(\mathbf{i}) \simeq \mathbf{k}_{\bar{d}}[x]e(\phi(\mathbf{i}))/J(\phi(\mathbf{i})), \quad x_r e(\mathbf{i}) \mapsto x_{r'} e(\phi(\mathbf{i})).$$

We will always identify $\mathbf{k}_\alpha^{(I)}$ with $\mathbf{k}_{\bar{\alpha}, \text{ord}}^{(\bar{I})}/J_{\bar{\alpha}, \text{ord}}$ via this isomorphism.

Lemma 2.10. *The action of the elements $e(\mathbf{i})$, $x_r e(\mathbf{i})$, $\tau_r e(\mathbf{i})$ of $R_{\alpha, \mathbf{k}}$ on $\mathbf{k}_\alpha^{(I)}$ is the same as the action of the elements $e(\phi(\mathbf{i}))$, $x_r^* e(\phi(\mathbf{i}))$, $\tau_r^* e(\phi(\mathbf{i}))$ of $S_{\bar{\alpha}, \mathbf{k}}$ on $\mathbf{k}_{\bar{\alpha}, \text{ord}}^{(\bar{I})}/J_{\bar{\alpha}, \text{ord}}$.*

Proof. The proof is based on the observation that by construction for each $i \in I_1$ and $j \in I_0$ we have

$$P_{i^1, j^0}(u, v) P_{i^2, j^0}(u, v) = P_{i, j}(u, v), \quad (8)$$

$$P_{j^0, i^1}(u, v)P_{j^0, i^2}(u, v) = P_{j, i}(u, v). \quad (9)$$

For each $\mathbf{i} \in I^\alpha$, we write $\phi(\mathbf{i}) = (i'_1, i'_2, \dots, i'_d)$. The only difficult part concerns the operator $\tau_r e(\mathbf{i})$ when at least one of the elements i_r, i_{r+1} is in I_1 . Assume that $i_r \in I_1, i_{r+1} \in I_0$. In this case we have

$$i'_{r'} = (i_r)^1 \in \bar{I}_1, \quad i'_{r'+1} = (i_r)^2 \in \bar{I}_2, \quad i'_{r'+2} = (i_{r+1})^0 \in \bar{I}_0.$$

In particular, the element $i'_{r'+2}$ is different from $i'_{r'}$ and $i'_{r'+1}$. Then, by (7), for each $f \in \mathbf{k}_{\bar{d}}[x]$ the element $\tau_r^* e(\phi(\mathbf{i})) = \tau_{r'} \tau_{r'+1} e(\phi(\mathbf{i}))$ maps $f e(\phi(\mathbf{i})) \in \mathbf{k}_{\bar{\alpha}, \text{ord}}^{(\bar{I})} / J_{\bar{\alpha}, \text{ord}}$ to

$$\begin{aligned} & P_{i'_{r'}, i'_{r'+2}}(x_{r'+1}, x_{r'}) s_{r'} \left(P_{i'_{r'+1}, i'_{r'+2}}(x_{r'+2}, x_{r'+1}) s_{r'+1}(f) \right) e(s_{r'} s_{r'+1}(\phi(\mathbf{i}))) = \\ &= P_{i'_{r'}, i'_{r'+2}}(x_{r'+1}, x_{r'}) P_{i'_{r'+1}, i'_{r'+2}}(x_{r'+2}, x_{r'}) s_{r'} s_{r'+1}(f) e(\phi(s_r(\mathbf{i}))) \\ &= P_{i_r, i_{r+1}}(x_{r'+1}, x_{r'}) s_{r'} s_{r'+1}(f) e(\phi(s_r(\mathbf{i}))), \end{aligned}$$

where the last equality holds by (8). Thus we see that the action of $\tau_r^* e(\phi(\mathbf{i}))$ on the polynomial representation is the same as the action of $\tau_r e(\mathbf{i})$. The case when $i_r \in I_0, i_{r+1} \in I_1$ can be done similarly.

Assume now that $i_r \neq i_{r+1}$ are both in I_1 . By the assumption on the quiver Γ (see Section 2.2), there are no arrows in Γ between i_r and i_{r+1} . Thus there are no arrows in $\bar{\Gamma}$ between any of the vertices $(i_r)^1 = i'_{r'}$ or $(i_r)^2 = i'_{r'+1}$ and any of the vertices $(i_{r+1})^1 = i'_{r'+2}$ or $(i_{r+1})^2 = i'_{r'+3}$. Then, by (7), for each $f \in \mathbf{k}_{\bar{d}}[x]$ the element $\tau_r^* e(\mathbf{i}) = \tau_{r'+1} \tau_{r'+2} \tau_{r'} \tau_{r'+1} e(\phi(\mathbf{i}))$ maps $f e(\phi(\mathbf{i}))$ to

$$s_{r'+1} s_{r'+2} s_{r'} s_{r'+1}(f) e(\phi(s_r(\mathbf{i}))).$$

Thus we see that the action of $\tau_r^* e(\phi(\mathbf{i}))$ on the polynomial representation is the same as the action of $\tau_r e(\mathbf{i})$.

Finally, assume that $i_r = i_{r+1} \in I_1$. In this case we have

$$(i_r)^1 = i'_{r'} = (i_{r+1})^1 = i'_{r'+2}, \quad (i_r)^2 = i'_{r'+1} = (i_{r+1})^2 = i'_{r'+3}.$$

Then, by (7), for each $f \in \mathbf{k}_{\bar{d}}[x]$ the element $\tau_r^* e(\phi(\mathbf{i})) = -\tau_{r'+1} \tau_{r'+2} \tau_{r'} \tau_{r'+1} e(\phi(\mathbf{i}))$ maps $f e(\phi(\mathbf{i}))$ to

$$s_{r'+1} \partial_{r'+2} \partial_{r'}(x_{r'+1} - x_{r'+2}) s_{r'+1}(f) e(\phi(s_r(\mathbf{i}))).$$

To prove that this gives the same result as for $\tau_r e(\mathbf{i})$, it is enough to check this on monomials $x_r^n x_{r+1}^m e(\mathbf{i})$. Assume for simplicity that $n \geq m$. The situation $n \leq m$ can be treated similarly. The element $\tau_r e(\mathbf{i})$ maps this monomial to

$$\partial_r(x_r^n x_{r+1}^m) e(\mathbf{i}) = - \sum_{a=m}^{n-1} x_r^a x_{r+1}^{n+m-1-a} e(\mathbf{i}).$$

Here the symbol $\sum_{a=x}^y$ means 0 when $y = x - 1$. The element $\tau_r^* e(\phi(\mathbf{i}))$ maps $x_{r'+1}^n x_{r'+2}^m e(\phi(\mathbf{i}))$ to

$$\begin{aligned}
& s_{r'+1} \partial_{r'+2} \partial_{r'} [x_{r'+1}^{m+1} x_{r'+2}^n - x_{r'+1}^m x_{r'+2}^{n+1}] e(\phi(\mathbf{i})) = \\
&= s_{r'+1} \left(-(\sum_{a=0}^m x_{r'+1}^a x_{r'+2}^{m-a}) (\sum_{b=0}^{n-1} x_{r'+2}^b x_{r'+3}^{n-1-b}) \right. \\
&\quad \left. + (\sum_{a=0}^{m-1} x_{r'+1}^a x_{r'+2}^{m-1-a}) (\sum_{b=0}^n x_{r'+2}^b x_{r'+3}^{n-b}) \right) e(\phi(\mathbf{i})) \\
&= \left[-(\sum_{a=0}^m x_{r'+1}^a x_{r'+2}^{m-a}) (\sum_{b=0}^{n-1} x_{r'+1}^b x_{r'+3}^{n-1-b}) \right. \\
&\quad \left. + (\sum_{a=0}^{m-1} x_{r'+1}^a x_{r'+2}^{m-1-a}) (\sum_{b=0}^n x_{r'+1}^b x_{r'+3}^{n-b}) \right] e(\phi(\mathbf{i})) \\
&= \left[-x_{r'+1}^m (\sum_{b=0}^{n-1} x_{r'+1}^b x_{r'+3}^{n-1-b}) \right. \\
&\quad \left. + x_{r'+1}^n (\sum_{a=0}^{m-1} x_{r'+1}^a x_{r'+2}^{m-1-a}) \right] e(\phi(\mathbf{i})) \\
&= \left[-x_{r'+1}^m (\sum_{b=0}^{n-1} x_{r'+1}^b x_{r'+2}^{n-1-b}) \right. \\
&\quad \left. + x_{r'+1}^n (\sum_{a=0}^{m-1} x_{r'+1}^a x_{r'+2}^{m-1-a}) \right] e(\phi(\mathbf{i})) \\
&= - \left[\sum_{a=m}^{n-1} x_{r'+1}^a x_{r'+2}^{m+n-1-a} \right] e(\phi(\mathbf{i})).
\end{aligned}$$

□

2.7 Isomorphism Φ

Theorem 2.11. *For each $\alpha \in Q_I^+$, there is an algebra isomorphism $\Phi_{\alpha, \mathbf{k}}: R_{\alpha, \mathbf{k}} \rightarrow S_{\bar{\alpha}, \mathbf{k}}$ such that*

$$\begin{aligned}
e(\mathbf{i}) &\mapsto e(\phi(\mathbf{i})), \\
x_r e(\mathbf{i}) &\mapsto x_r^* e(\phi(\mathbf{i})), \\
\tau_r e(\mathbf{i}) &\mapsto \tau_r^* e(\phi(\mathbf{i})).
\end{aligned}$$

Proof. In view of Lemma 2.10, it is enough to prove two following facts:

- the elements $e(\phi(\mathbf{i})), x_r^*, \tau_r^*$ generate $S_{\bar{\alpha}, \mathbf{k}}$,
- the representation $\mathbf{k}_{\bar{\alpha}, \text{ord}}^{(\bar{I})} / J_{\bar{\alpha}, \text{ord}}$ of $S_{\bar{\alpha}, \mathbf{k}}$ is faithful.

Fix $\mathbf{i}, \mathbf{j} \in I^\alpha$. Set $\mathbf{i}' = (i'_1, \dots, i'_d) = \phi(\mathbf{i})$, $\mathbf{j}' = \phi(\mathbf{j})$. Let \mathbf{B} (resp. \mathbf{B}') be the basis of $e(\mathbf{j}) R_{\alpha, \mathbf{k}} e(\mathbf{i})$ (resp. $e(\mathbf{j}') R_{\bar{\alpha}, \mathbf{k}} e(\mathbf{i}')$) as in Remark 2.2. For each element $b = \tau_w x_1^{a_1} \cdots x_d^{a_d} e(\mathbf{i}) \in \mathbf{B}$ we construct an element $b^* \in e(\mathbf{j}') S_{\bar{\alpha}, \mathbf{k}} e(\mathbf{i}')$ that acts by the same operator on the polynomial representation. Recall that the definition of b depends on the choice of a reduced expression $w = s_{p_1} \cdots s_{p_k}$. We set

$$b^* = \tau_{p_1}^* \cdots \tau_{p_k}^* (x_1^*)^{a_1} \cdots (x_d^*)^{a_d} e(\mathbf{i}') \in e(\mathbf{j}') S_{\bar{\alpha}, \mathbf{k}} e(\mathbf{i}').$$

Let us call the permutation $w \in \mathfrak{S}_{\mathbf{i}', \mathbf{j}'}$ *balanced* if we have $w(a+1) = w(a)+1$ for each a such that $i'_a = i^1$ for some $i \in I$ (and thus $i'_{a+1} = i^2$). Otherwise we say that w is *unbalanced*. There exists a unique map $u: \mathfrak{S}_{\mathbf{i}, \mathbf{j}} \rightarrow \mathfrak{S}_{\mathbf{i}', \mathbf{j}'}$ such that

for each $w \in \mathfrak{S}_{\mathbf{i}, \mathbf{j}}$ the permutation $u(w)$ is balanced and $w(r) < w(t)$ if and only if $u(w)(r') < u(w)(t')$ for each $r, t \in [1, d]$, where $r' = r'_{\mathbf{i}}$ and $t' = t'_{\mathbf{i}}$ are as in Section 2.6. The image of u is exactly the set of all balanced permutations in $\mathfrak{S}_{\mathbf{i}', \mathbf{j}'}$.

Assume that $w \in \mathfrak{S}_{\mathbf{i}', \mathbf{j}'}$ is unbalanced. We claim that there exists an index a such that $i'_a \in \bar{I}_1$ and $w(a) > w(a+1)$. Really, let J be the set of indices $a \in [1, \bar{d}]$ such that $i'_a \in \bar{I}_1$. As \mathbf{j}' is well-ordered, we have $\sum_{a \in J} (w(a+1) - w(a)) = \#J$. As w is unbalanced, not all summands in this sum are equal to 1. Then one of the summands must be negative. Let $a \in J$ be an index such that $w(a) > w(a+1)$. We can assume that the reduced expression of w is of the form $w = s_{p_1} \cdots s_{p_k} s_a$. In this case the element $\tau_w e(\mathbf{i}')$ is zero in $S_{\bar{\alpha}, \mathbf{k}}$ because the idempotent $s_a(\mathbf{i}')$ is unordered.

Assume that $w \in \mathfrak{S}_{\mathbf{i}', \mathbf{j}'}$ is balanced. Thus there exists some $\tilde{w} \in \mathfrak{S}_{\mathbf{i}, \mathbf{j}}$ such that $u(\tilde{w}) = w$. We choose an arbitrary reduced expression $\tilde{w} = s_{p_1} \cdots s_{p_k}$ and we choose the reduced expression of $w = s_{q_1} \cdots s_{q_r}$ induced by the reduced expression of \tilde{w} . Then the element $\tau_w e(\mathbf{i}') = \tau_{q_1} \cdots \tau_{q_r} e(\mathbf{i}') \in S_{\alpha, \mathbf{k}}$ is equal to $\pm(\tau_{p_1} \cdots \tau_{p_k} e(\mathbf{i}))^*$.

The discussion above shows that the image of an element $b' \in \mathbf{B}'$ in $e(\mathbf{j}') S_{\bar{\alpha}, \mathbf{k}} e(\mathbf{i}')$ is either zero or is of the form $\pm b^*$ for some $b \in \mathbf{B}$. Moreover, each b^* for $b \in \mathbf{B}$ can be obtained in such a way. Now we get the following.

- The elements $e(\phi(\mathbf{i}))$, x_r^* , τ_r^* generate $S_{\bar{\alpha}, \mathbf{k}}$ because the image of each element of \mathbf{B}' in $e(\mathbf{j}') S_{\bar{\alpha}, \mathbf{k}} e(\mathbf{i}')$ is either zero or a monomial on $e(\phi(\mathbf{i}))$, x_r^* , τ_r^* .
- The representation $\mathbf{k}_{\bar{\alpha}, \text{ord}}^{(\bar{I})} / J_{\bar{\alpha}, \text{ord}}$ of $S_{\bar{\alpha}, \mathbf{k}}$ is faithful because the spanning set $\{b^*; b \in \mathbf{B}\}$ of $e(\mathbf{j}') S_{\bar{\alpha}, \mathbf{k}} e(\mathbf{i}')$ acts on the polynomial representation by linearly independent operators (because the polynomial representation of $R_{\alpha, \mathbf{k}}$ is faithful).

□

The center of the algebra $R_{\alpha, \mathbf{k}}$ is the ring of symmetric polynomials $\mathbf{k}_d[x]^{\mathfrak{S}_d}$, see [17, Prop. 3.9]. Thus $S_{\bar{\alpha}, \mathbf{k}}$ is a $\mathbf{k}_d[x]^{\mathfrak{S}_d}$ -algebra under the isomorphism $\Phi_{\alpha, \mathbf{k}}$. Let Σ be the polynomial $\prod_{a < b} (x_a - x_b)^2 \in \mathbf{k}_d[x]^{\mathfrak{S}_d}$. Let $R_{\alpha, \mathbf{k}}[\Sigma^{-1}]$, $S_{\bar{\alpha}, \mathbf{k}}[\Sigma^{-1}]$ be the rings of quotients of $R_{\alpha, \mathbf{k}}$, $S_{\bar{\alpha}, \mathbf{k}}$ obtained by inverting Σ . We can extend the isomorphism $\Phi_{\alpha, \mathbf{k}}$ from Theorem 2.11 to an algebra isomorphism $\Phi_{\alpha, \mathbf{k}}: R_{\alpha, \mathbf{k}}[\Sigma^{-1}] \rightarrow S_{\bar{\alpha}, \mathbf{k}}[\Sigma^{-1}]$.

From now on, we assume that the connected components of the quiver Γ are of the form Γ_a for $a \in \mathbb{N}$, $a > 1$ or $a = \infty$. Note that there is an action of the symmetric group \mathfrak{S}_d on $\mathbf{k}_d^{(I)}$ permuting the variables and the components of \mathbf{i} . Consider the following elements in $R_{\alpha, \mathbf{k}}[\Sigma^{-1}]$:

$$\Psi_r e(\mathbf{i}) = \begin{cases} ((x_r - x_{r+1})\tau_r + 1)e(\mathbf{i}) & \text{if } i_{r+1} = i_r, \\ -(x_r - x_{r+1})^{-1}\tau_r e(\mathbf{i}) & \text{if } i_{r+1} = i_r - 1, \\ \tau_r e(\mathbf{i}) & \text{else.} \end{cases}$$

The element $\Psi_r e(\mathbf{i})$ is called *intertwining operator*. Using the formulas (7) we can check that $\Psi_r e(\mathbf{i})$ still acts on the polynomial representation and the corresponding operator is equal to $s_r e(\mathbf{i})$. Note also that $\tilde{\Psi}_r = (x_r - x_{r+1})\Psi_r$ is an element of $R_{\alpha, \mathbf{k}}$.

Lemma 2.12. *The images of intertwining operators by the morphism $\Phi_{\alpha, \mathbf{k}}: R_{\alpha, \mathbf{k}} \rightarrow S_{\overline{\alpha}, \mathbf{k}}$ can be described in the following way. For $\mathbf{i} \in I^\alpha$ such that $i_r - 1 \neq i_{r+1}$ we have*

$$\Phi_{\alpha, \mathbf{k}}(\Psi_r e(\mathbf{i})) = \begin{cases} \Psi_{r'} e(\phi(\mathbf{i})), & \text{if } i_r, i_{r+1} \in I_0, \\ \Psi_{r'} \Psi_{r'+1} e(\phi(\mathbf{i})) & \text{if } i_r \in I_1, i_{r+1} \in I_0, \\ \Psi_{r'+1} \Psi_{r'} e(\phi(\mathbf{i})) & \text{if } i_r \in I_0, i_{r+1} \in I_1, \\ \Psi_{r'+1} \Psi_{r'+2} \Psi_{r'} \Psi_{r'+1} e(\phi(\mathbf{i})) & \text{if } i_r, i_{r+1} \in I_1. \end{cases}$$

For $\mathbf{i} \in I^\alpha$ such that $i_r - 1 = i_{r+1}$ we have

$$\Phi_{\alpha, \mathbf{k}}(\tilde{\Psi}_r e(\mathbf{i})) = \begin{cases} \tilde{\Psi}_{r'} e(\phi(\mathbf{i})), & \text{if } i_r, i_{r+1} \in I_0, \\ \tilde{\Psi}_{r'} \tilde{\Psi}_{r'+1} e(\phi(\mathbf{i})) & \text{if } i_r \in I_1, i_{r+1} \in I_0, \\ \tilde{\Psi}_{r'+1} \tilde{\Psi}_{r'} e(\phi(\mathbf{i})) & \text{if } i_r \in I_0, i_{r+1} \in I_1. \end{cases}$$

Here $r' = r'_1$ is as in Section 2.6.

Proof. The right hand side in the formulas for $\Phi_{\alpha, \mathbf{k}}(\Psi_r e(\mathbf{i}))$ is an element X in $S_{\overline{\alpha}, \mathbf{k}}[\Sigma^{-1}]$ that acts by the same operator that $\Psi_r e(\mathbf{i})$ on the polynomial representation. However, since the polynomial representation of $S_{\overline{\alpha}, \mathbf{k}}$ is faithful by Lemma 2.9, and since there is a well-defined element of $S_{\overline{\alpha}, \mathbf{k}}$ which acts in the same way as X in this polynomial representation, we conclude that X makes sense as an element of $S_{\overline{\alpha}, \mathbf{k}}$. \square

The goal of the rest of Section 2 is to obtain a deformed version of the isomorphism from Theorem 2.11 over some local ring R (see Section 2.9). More precisely, the localized Hecke algebra over a field is isomorphic to the localized KLR algebra (see Section 2.11). We want to construct an R -algebra homomorphism between a localized Hecke algebra and a balanced analogue of a localized Hecke algebra (see Section 2.12 for the choice of parameters and Section 2.13 for the definition of a balanced analogue of the cyclotomic Hecke algebra) such that this homomorphism is compatible with the localization of the homomorphism from Theorem 2.11 over the residue field of R and over its field of fractions.

2.8 Special quivers

From now on we will be interested only in some special types of quivers.

First, consider the quiver $\Gamma = \Gamma_e$, where e is an integer > 1 . In particular, from now on we fix $I = \mathbb{Z}/e\mathbb{Z}$. Fix $k \in [0, e-1]$ and set $I_1 = \{k\}$, $I_0 = I \setminus \{k\}$. In this case the quiver $\overline{\Gamma}$ is isomorphic to Γ_{e+1} . More precisely, the decomposition $\overline{I} = \overline{I}_0 \sqcup \overline{I}_2 \sqcup \overline{I}_2$ is such that $\overline{I}_1 = \{k\}$, $\overline{I}_2 = \{k+1\}$. To avoid confusion, for $i \in \overline{I}$ we will write $\overline{\alpha}_i$, $\overline{\varepsilon}_i$ and $\overline{\Lambda}_i$ for α_i , ε_i and Λ_i respectively.

Remark 2.13. If Γ is as above, a sequence $\mathbf{i} = (i_1, \dots, i_d) \in \overline{I}^d$ is well-ordered if for each index a such that $i_a = k$ we have $a < d$ and $i_{a+1} = k+1$. The sequence \mathbf{i} is unordered if there is $r \leq d$ such that the subsequence (i_1, \dots, i_r) contains more elements equal to $k+1$ than elements equal to k .

Let $\Upsilon: \mathbb{Z} \rightarrow \mathbb{Z}$ be the map given for $a \in \mathbb{Z}, b \in [0, e-1]$ by

$$\Upsilon(ae+b) = \begin{cases} a(e+1)+b & \text{if } b \in [0, k], \\ a(e+1)+b+1 & \text{if } b \in [k+1, e-1]. \end{cases} \quad (10)$$

Now, consider the quiver $\tilde{\Gamma} = (\Gamma_\infty)^{\oplus l}$. Set $\tilde{\Gamma} = (\tilde{I}, \tilde{H})$ and write $\tilde{\alpha}_i, \tilde{\varepsilon}_i$ and $\tilde{\Lambda}_i$ for α_i, ε_i and Λ_i respectively for each $i \in \tilde{I}$. We identify an element of \tilde{I} with an element $(a, b) \in I \times [1, l]$ in the obvious way. Consider the decomposition $\tilde{I} = \tilde{I}_0 \sqcup \tilde{I}_1$ such that $(a, b) \in \tilde{I}_1$ if and only if $a \equiv k \pmod{e}$. In this case the quiver $\tilde{\Gamma}$ is isomorphic to $\tilde{\Gamma}$. More precisely, in this case we have

$$\begin{aligned} (a, b)^0 &= (\Upsilon(a), b), \\ (a, b)^1 &= (\Upsilon(a), b), \\ (a, b)^2 &= (\Upsilon(a) + 1, b). \end{aligned}$$

To distinguish notations, we will always write $\tilde{\phi}$ for any of the maps $\tilde{\phi}: \tilde{I}^\infty \rightarrow \tilde{I}^\infty$, $Q_{\tilde{I}} \rightarrow Q_{\tilde{I}}, X_{\tilde{I}} \rightarrow X_{\tilde{I}}$ in Section 2.2.

From now on we write $\Gamma = \Gamma_e, \bar{\Gamma} = \Gamma_{e+1}$ and $\tilde{\Gamma} = (\Gamma_\infty)^{\oplus l}$. Recall that

$$I = I_e = \mathbb{Z}/e\mathbb{Z}, \quad \bar{I} = I_{e+1} = \mathbb{Z}/(e+1)\mathbb{Z}, \quad \tilde{I} = (I_\infty)^{\oplus l} = \mathbb{Z} \times [1, l].$$

Consider the quiver homomorphism $\pi_e: \tilde{\Gamma} \rightarrow \Gamma$ such that

$$\pi_e: \tilde{I} \rightarrow I, \quad (a, b) \mapsto a \pmod{e}.$$

Then π_{e+1} is a quiver homomorphism $\pi_{e+1}: \tilde{\Gamma} \rightarrow \bar{\Gamma}$. They yield \mathbb{Z} -linear maps

$$\pi_e: Q_{\tilde{I}} \rightarrow Q_I, \quad \pi_e: X_{\tilde{I}} \rightarrow X_I, \quad \pi_{e+1}: Q_{\tilde{I}} \rightarrow Q_{\bar{I}}, \quad \pi_{e+1}: X_{\tilde{I}} \rightarrow X_{\bar{I}}.$$

The following diagrams are commutative for $\alpha \in Q_I^+, \tilde{\alpha} \in Q_{\tilde{I}}^+$ such that $\pi_e(\tilde{\alpha}) = \alpha$,

$$\begin{array}{ccccc} Q_{\tilde{I}} & \xrightarrow{\tilde{\phi}} & Q_{\tilde{I}} & X_{\tilde{I}} & \xrightarrow{\tilde{\phi}} & X_{\tilde{I}} & \tilde{I}^\alpha & \xrightarrow{\tilde{\phi}} & \tilde{I}^{\phi(\tilde{\alpha})} \\ \pi_e \downarrow & & \pi_{e+1} \downarrow & \pi_e \downarrow & & \pi_{e+1} \downarrow & \pi_e \downarrow & & \pi_{e+1} \downarrow \\ Q_I & \xrightarrow{\phi} & Q_{\bar{I}} & X_I & \xrightarrow{\phi} & X_{\bar{I}} & I^\alpha & \xrightarrow{\phi} & \bar{I}^{\phi(\alpha)} \end{array}$$

2.9 Deformation rings

In this section we introduce some general definitions from [19] for a later use.

We call *deformation ring* $(R, \kappa, \tau_1, \dots, \tau_l)$ a regular commutative noetherian \mathbb{C} -algebra R with 1 equipped with a homomorphism $\mathbb{C}[\kappa^{\pm 1}, \tau_1, \dots, \tau_l] \rightarrow R$. Let $\kappa, \tau_1, \dots, \tau_r$ denote also the images of $\kappa, \tau_1, \dots, \tau_r$ in R . A deformation ring is *in general position* if any two elements of the set

$$\{\tau_u - \tau_v + a\kappa + b, \kappa - c; a, b \in \mathbb{Z}, c \in \mathbb{Q}, u \neq v\}$$

have no common non-trivial divisors. A *local deformation ring* is a deformation ring which is a local ring and such that $\tau_1, \dots, \tau_l, \kappa - e$ belong to the maximal ideal of R . Note that each \mathbb{C} -algebra that is a field has a *trivial* local deformation ring structure, i.e., such that $\tau_1 = \dots = \tau_l = 0$ and $\kappa = e$. We always consider \mathbb{C} as a local deformation ring with a trivial deformation ring structure. A \mathbb{C} -algebra R is called *analytic* if it is a localization of the ring of germs of holomorphic functions on some compact polydisc $D \subset \mathbb{C}^d$ for some $d \geq 1$.

We will write $\bar{\kappa} = \kappa(e+1)/e$, $\bar{\tau}_r = \tau_r(e+1)/e$. We will abbreviate R for $(R, \kappa, \tau_1, \dots, \tau_l)$ and \bar{R} for $(R, \bar{\kappa}, \bar{\tau}_1, \dots, \bar{\tau}_l)$.

Let R be a local analytic deformation ring with residue field \mathbf{k} . Consider the elements $q_e = \exp(2\pi\sqrt{-1}/\kappa)$ and $q_{e+1} = \exp(2\pi\sqrt{-1}/\bar{\kappa})$ in R . These elements specialize to $\zeta_e = \exp(2\pi\sqrt{-1}/e)$ and $\zeta_{e+1} = \exp(2\pi\sqrt{-1}/(e+1))$ in \mathbf{k} .

2.10 Hecke algebras

Let R be a commutative ring with 1. Fix an element $q \in R$.

Definition 2.14. The *affine Hecke algebra* $H_{R,d}(q)$ is the R -algebra generated by T_1, \dots, T_{d-1} and the invertible elements X_1, \dots, X_d modulo the following defining relations

$$\begin{aligned} T_r T_s &= T_s T_r \text{ if } |r-s| > 1, & T_r T_{r+1} T_r &= T_{r+1} T_r T_{r+1} \\ (T_r - q)(T_r + 1) &= 0, & T_r X_r &= X_r T_r \text{ if } |r-s| > 1, \\ T_r X_{r+1} &= X_r T_r + (q-1)X_{r+1}, & T_r X_r &= X_{r+1} T_r - (q-1)X_{r+1}. \end{aligned}$$

Fix elements $Q_1, \dots, Q_l \in R$.

Definition 2.15. The *cyclotomic Hecke algebra* $H_{d,R}^Q(q)$ is the quotient of $H_{d,R}(q)$ by the two-sided ideal generated by $(X_1 - Q_1) \cdots (X_l - Q_l)$.

Assume that $R = \mathbf{k}$ is a field and $q \neq 0, 1$, $Q_r \neq 0$. The algebra $H_{d,\mathbf{k}}(q)$ has a faithful representation in the vector space $\mathbf{k}[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ such that $X_r^{\pm 1}$ acts by multiplication by $X_r^{\pm 1}$ and T_r by

$$T_r(P) = qs_r(P) + (q-1)X_{r+1}(X_r - X_{r+1})^{-1}(s_r(P) - P).$$

The following operator acts on $\mathbf{k}[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ as the reflection s_r

$$\Psi_r = \frac{X_r - X_{r+1}}{qX_r - X_{r+1}}(T_r - q) + 1 = (T_r + 1) \frac{X_r - X_{r+1}}{X_r - qX_{r+1}} - 1.$$

For a future use, consider the element $\tilde{\Psi}_r \in H_{d,\mathbf{k}}$ given by

$$\tilde{\Psi}_r = (qX_r - X_{r+1})\Psi_r = (X_r - X_{r+1})T_r + (q-1)X_{r+1}.$$

Now, consider the subset $\mathcal{F} = \mathcal{F}(q, Q)$ of \mathbf{k} given by

$$\mathcal{F}(q, Q) = \bigcup_{r \in \mathbb{Z}, t \in [1, l]} \{q^r Q_t\}. \quad (11)$$

We can consider \mathcal{F} as the vertex set of a quiver with an arrow $i \rightarrow j$ if and only if $j = qi$. Let M be a finite dimensional $H_{d,\mathbf{k}}^Q(q)$ -module. For each $\mathbf{i} = (i_1, \dots, i_d) \in \mathcal{F}^d$ let $M_{\mathbf{i}}$ be the generalized eigenspace of X_1, \dots, X_d with eigenvalues i_1, \dots, i_d . The algebra $H_{d,\mathbf{k}}^Q(q)$ contains an idempotent $e(\mathbf{i})$ which acts on each finite dimensional $H_{d,\mathbf{k}}^Q(q)$ -module M by projection to $M_{\mathbf{i}}$ with respect to $\bigoplus_{\mathbf{j} \neq \mathbf{i}} M_{\mathbf{j}}$. For $\alpha \in Q_{\mathcal{F}}^+$ we consider the central idempotent $e(\alpha) = \sum_{\mathbf{i} \in \mathcal{F}^\alpha} e(\mathbf{i})$. The algebra $H_{d,\mathbf{k}}^Q$ decomposes in a direct sum of blocks $H_{\alpha,\mathbf{k}}^Q$ with $\alpha \in Q_{\mathcal{F}}^+$ where $H_{\alpha,\mathbf{k}}^Q = e(\alpha)H_{d,\mathbf{k}}^Q$. See [4] for more details.

2.11 The isomorphism between Hecke and KLR algebras

Assume that $R = \mathbf{k}$ is a field and $q \neq 0, 1$.

First, we define some localized versions of Hecke algebras and KLR algebras. Let \mathcal{F} be a subset of \mathbf{k} such that $q^{\mathbb{Z}}\mathcal{F}/q^{\mathbb{Z}}$ is finite. We view \mathcal{F} as the vertex set of a quiver with an arrow $i \rightarrow j$ if and only if $j = qi$. Consider the algebra

$$A_1 = \bigoplus_{\mathbf{i} \in \mathcal{F}^d} \mathbf{k}[X_1^{\pm 1}, \dots, X_d^{\pm 1}][(X_r - X_t)^{-1}, (qX_r - X_t)^{-1}; r \neq t]e(\mathbf{i}),$$

where $e(\mathbf{i})$ are orthogonal idempotents and X_r commutes with $e(\mathbf{i})$. Let $H_{d,\mathbf{k}}^{\text{loc}}(q)$ be the A_1 -module given by the extension of scalars from the $\mathbf{k}[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$ -module $H_{d,\mathbf{k}}(q)$. It is an A_1 -algebra such that

$$T_r e(\mathbf{i}) - e(s_r(\mathbf{i}))T_r = (1 - q)X_{r+1}(X_r - X_{r+1})^{-1}(e(\mathbf{i}) - e(s_r(\mathbf{i})))$$

and

$$Z^{-1}T_r = T_r Z^{-1}, \quad \text{where } Z = \prod_{r < t} (X_r - X_t)^2 \prod_{r \neq t} (qX_r - X_t)^2.$$

In this section the KLR algebras are always defined with respect to the quiver \mathcal{F} . We consider the algebra

$$A_2 = \bigoplus_{\mathbf{i} \in \mathcal{F}^d} \mathbf{k}[x_1, \dots, x_d][S_{\mathbf{i}}^{-1}]e(\mathbf{i}),$$

where

$$S_{\mathbf{i}} = \{(x_r + 1), (i_r(x_r + 1) - i_t(x_t + 1)), (qi_r(x_r + 1) - i_t(x_t + 1)); r \neq t\}.$$

There is a \mathbf{k} -algebra structure on $R_{d,\mathbf{k}}^{\text{loc}} = A_2 \otimes_{\mathbf{k}_d^{(\mathcal{F})}} R_{d,\mathbf{k}}$, where $\mathbf{k}_d^{(\mathcal{F})}$ is as in (6).

The polynomial representations of $H_{d,\mathbf{k}}(q)$ and $R_{d,\mathbf{k}}$ yield faithful representations of $H_{d,\mathbf{k}}^{\text{loc}}(q)$ and $R_{d,\mathbf{k}}^{\text{loc}}$ on A_1 and A_2 respectively. Moreover, there is an isomorphism of \mathbf{k} -algebras $A_2 \simeq A_1$ given by $x_re(\mathbf{i}) \mapsto (i_r^{-1}X_r - 1)e(\mathbf{i})$. This yields the following proposition.

Proposition 2.16. *There is an isomorphism of \mathbf{k} -algebras $R_{d,\mathbf{k}}^{\text{loc}} \simeq H_{d,\mathbf{k}}^{\text{loc}}(q)$ such that*

$$\begin{aligned} e(\mathbf{i}) &\mapsto e(\mathbf{i}), \\ x_re(\mathbf{i}) &\mapsto (i_r^{-1}X_r - 1)e(\mathbf{i}), \\ \Psi_re(\mathbf{i}) &\mapsto \Psi_re(\mathbf{i}). \end{aligned}$$

□

Now, we consider the subalgebra $\widehat{R}_{d,\mathbf{k}}$ of $R_{d,\mathbf{k}}^{\text{loc}}$ generated by

- the elements of $R_{d,\mathbf{k}}$,
- the elements $(x_r + 1)^{-1}$,
- the elements of the form $(i_r(x_r + 1) - i_t(x_t + 1))^{-1}e(\mathbf{i})$ such that $r \neq t$, $i_r \neq i_t$,
- the elements of the form $(qi_r(x_r + 1) - i_t(x_t + 1))^{-1}e(\mathbf{i})$ such that $r \neq t$, $qi_r \neq i_t$.

Similarly, consider the subalgebra $\widehat{H}_{d,\mathbf{k}}(q)$ of $H_{d,\mathbf{k}}^{\text{loc}}(q)$ generated by

- the elements of $H_{d,\mathbf{k}}(q)$,
- the elements of the form $(X_r - X_t)^{-1}e(\mathbf{i})$ such that $r \neq t$, $i_r \neq i_t$,
- the elements of the form $(qX_r - X_t)^{-1}e(\mathbf{i})$ such that $r \neq t$, $qi_r \neq i_t$.

Note that the element $\Psi_re(\mathbf{i}) \in H_{d,\mathbf{k}}^{\text{loc}}(q)$ belongs to $\widehat{H}_{d,\mathbf{k}}(q)$ if $i_r \neq qi_{r+1}$. We have the following lemma, see also [17, Sec. 3.2].

Proposition 2.17. *The isomorphism $R_{d,\mathbf{k}}^{\text{loc}} \simeq H_{d,\mathbf{k}}^{\text{loc}}(q)$ from Proposition 2.16 restricts to an isomorphism $\widehat{R}_{d,\mathbf{k}} \simeq \widehat{H}_{d,\mathbf{k}}(q)$.*

Fix $Q = (Q_1, \dots, Q_l)$, $Q_r \in \mathbf{k}$, $Q_r \neq 0$. Let $\mathcal{F} = \mathcal{F}(q, Q) \subset \mathbf{k}$ be as in (11). Consider the weight $\Lambda = \sum_{r=1}^l \Lambda_{Q_r} \in P_{\mathcal{F}}^+$.

Corollary 2.18. *There is an isomorphism between the cyclotomic KLR algebra $R_{d,\mathbf{k}}^\Lambda$ associated with the quiver \mathcal{F} and the cyclotomic Hecke algebra $H_{d,\mathbf{k}}^Q(q)$. Moreover, this isomorphism takes the block $R_{\alpha,\mathbf{k}}^\Lambda$ to the block $H_{\alpha,\mathbf{k}}^Q(q)$.*

Proof. The cyclotomic quotients $R_{d,\mathbf{k}} \rightarrow R_{d,\mathbf{k}}^\Lambda$ and $H_{d,\mathbf{k}}(q) \rightarrow H_{d,\mathbf{k}}^Q(q)$ factor through

$$\begin{array}{ccc} & \widehat{R}_{d,\mathbf{k}} & \\ \nearrow & & \searrow \\ R_{d,\mathbf{k}} & \xrightarrow{\quad} & R_{d,\mathbf{k}}^\Lambda \end{array} \quad \begin{array}{ccc} & \widehat{H}_{d,\mathbf{k}}(q) & \\ \nearrow & & \searrow \\ H_{d,\mathbf{k}}(q) & \xrightarrow{\quad} & H_{d,\mathbf{k}}^Q(q) \end{array}$$

because the elements $(i_r(x_r + 1) - i_r(x_r + 1))e(\mathbf{i})$, for $r \neq t$, $i_r \neq i_t$, etc., are invertible in $e(\mathbf{i})R_{d,\mathbf{k}}^\Lambda e(\mathbf{i})$ (or resp. $e(\mathbf{i})H_{d,\mathbf{k}}^Q(q)e(\mathbf{i})$). Thus the isomorphism $\widehat{R}_{d,\mathbf{k}} \simeq \widehat{H}_{d,\mathbf{k}}(q)$ yields an isomorphism $R_{d,\mathbf{k}}^\Lambda \simeq H_{d,\mathbf{k}}^Q(q)$. \square

2.12 The choice of the parameters

Let R be a local deformation ring with residue field \mathbf{k} and field of fractions K . Fix a tuple $\nu = (\nu_1, \dots, \nu_l) \in \mathbb{Z}^l$. Put $q = q_e$ and $Q_r = \exp(2\pi\sqrt{-1}(\nu_r + \tau_r)/\kappa)$ for $r \in [1, l]$. The canonical morphism $R \rightarrow \mathbf{k}$ maps q_e to ζ_e and Q_r to $\zeta_e^{\nu_r}$. Let $H_{d,R}^\nu(q_e)$ be the cyclotomic Hecke algebra over R with parameters q, Q_r . Set $H_{d,\mathbf{k}}^\nu(\zeta_e) = \mathbf{k} \otimes_R H_{d,R}^\nu(q_e)$ and $H_{d,K}^\nu(q_e) = K \otimes_R H_{d,R}^\nu(q_e)$.

Fix $k \in [0, e-1]$. To this k we associate a map $\Upsilon: \mathbb{Z} \rightarrow \mathbb{Z}$ as in (10). Consider the tuple

$$\overline{\nu} = (\overline{\nu}_1, \dots, \overline{\nu}_l) \in \mathbb{Z}^l, \quad \overline{\nu}_r = \Upsilon(\nu_r) \quad \forall r \in [1, l]. \quad (12)$$

Consider the cyclotomic Hecke algebra $H_{d,\overline{R}}^{\overline{\nu}}(q_{e+1})$ with parameters $q = q_{e+1}$, $\overline{Q} = (\overline{Q}_1, \dots, \overline{Q}_l)$, where $\overline{Q}_r = \exp(2\pi\sqrt{-1}(\overline{\nu}_r + \overline{\tau}_r)/\overline{\kappa})$ and $\overline{\kappa}, \overline{\tau}_r$ are defined in Section 2.9. Set

$$H_{d,\overline{\mathbf{k}}}^{\overline{\nu}}(\zeta_{e+1}) = \overline{\mathbf{k}} \otimes_{\overline{R}} H_{d,\overline{R}}^{\overline{\nu}}(q_{e+1}), \quad H_{d,\overline{K}}^{\overline{\nu}}(q_{e+1}) = \overline{K} \otimes_{\overline{R}} H_{d,\overline{R}}^{\overline{\nu}}(q_{e+1}).$$

2.13 The deformation of the isomorphism Φ

Let $\Gamma = (I, H)$, $\overline{\Gamma} = (\overline{I}, \overline{H})$ and $\tilde{\Gamma} = (\tilde{I}, \tilde{H})$ be as in Section 2.8. We have the following isomorphisms of quivers

$$\tilde{I} \simeq \mathcal{F}(q_e, Q), \quad i = (a, b) \mapsto p_i := \exp(2\pi\sqrt{-1}(a + \tau_b)/\kappa), \quad (13)$$

$$\tilde{I} \simeq \mathcal{F}(q_{e+1}, \overline{Q}), \quad i = (a, b) \mapsto \overline{p}_i := \exp(2\pi\sqrt{-1}(a + \overline{\tau}_b)/\overline{\kappa}), \quad (14)$$

where $\mathcal{F}(q, Q)$ is as in (11), Q and \overline{Q} are as in Section 2.12, $a \in \mathbb{Z}$, $b \in [1, l]$. Notice that $Q_r = p_{(\nu_r, r)}$ and $\overline{Q}_r = \overline{p}_{(\overline{\nu}_r, r)}$ for each r .

The map $R \rightarrow \mathbf{k}$ takes $\mathcal{F}(q_e, Q)$ to $\mathcal{F}(\zeta_e, Q)$ and $\mathcal{F}(q_{e+1}, \overline{Q})$ to $\mathcal{F}(\zeta_{e+1}, \overline{Q})$. We have the following isomorphisms of quivers

$$I \simeq \mathcal{F}(\zeta_e, Q), \quad i \mapsto p_i := \zeta_e^i, \quad (15)$$

$$\overline{I} \simeq \mathcal{F}(\zeta_{e+1}, \overline{Q}), \quad i \mapsto \overline{p}_i := \zeta_{e+1}^i. \quad (16)$$

These isomorphisms yield the following commutative diagrams

$$\begin{array}{ccc} \tilde{I} & \xrightarrow{\sim} & \mathcal{F}(q_e, Q) \\ \pi_e \downarrow & & \downarrow \\ I & \xrightarrow{\sim} & \mathcal{F}(\zeta_e, Q), \end{array} \quad \begin{array}{ccc} \tilde{I} & \xrightarrow{\sim} & \mathcal{F}(q_{e+1}, \overline{Q}) \\ \pi_{e+1} \downarrow & & \downarrow \\ \overline{I} & \xrightarrow{\sim} & \mathcal{F}(\zeta_{e+1}, \overline{Q}). \end{array}$$

We will identify

$$I \simeq \mathcal{F}(\zeta_e, Q), \quad \overline{I} \simeq \mathcal{F}(\zeta_{e+1}, \overline{Q}), \quad \tilde{I} \simeq \mathcal{F}(q_e, Q), \quad \tilde{I} \simeq \mathcal{F}(q_{e+1}, \overline{Q}) \quad (17)$$

as above.

Recall from Section 2.10 that the cyclotomic Hecke algebra $H_{d,\mathbf{k}}^\nu(\zeta_e)$ contains some idempotents $e(\mathbf{i})$ for $\mathbf{i} \in I^d$. These idempotents lift to idempotents in the R -algebra $H_{d,R}^\nu(q_e)$ because this algebra is free over R by [14, Thm. 2.2] and R is a local algebra, see, e.g., [6, Ex. 6.16]. We also denote these idempotents $e(\mathbf{i})$.

By base change we have the K -algebra $H_{d,K}^\nu(q_e)$ which contains some idempotents $e(\mathbf{j}) \in H_{d,K}^\nu(q_e)$ for $\mathbf{j} \in \tilde{I}^d$. The idempotent $e(\mathbf{i}) \in H_{d,R}^\nu(q_e)$ decomposes in $H_{d,K}^\nu(q_e)$ in the following way

$$e(\mathbf{i}) = \sum_{\mathbf{j} \in \tilde{I}^d, \pi_e(\mathbf{j})=\mathbf{i}} e(\mathbf{j}).$$

We have the decompositions

$$H_{d,R}^\nu(q_e) = \bigoplus_{\alpha \in Q_I^+, |\alpha|=d} H_{\alpha,R}^\nu(q_e),$$

$$H_{d,K}^\nu(q_e) = \bigoplus_{\alpha \in Q_I^+, |\alpha|=d} H_{\alpha,K}^\nu(q_e),$$

where

$$H_{\alpha,K}^\nu(q_e) = \bigoplus_{\tilde{\alpha} \in Q_I^+, \pi_e(\tilde{\alpha})=\alpha} H_{\tilde{\alpha},K}^\nu(q_e).$$

Our goal is to obtain an analogue of Theorem 2.11 over the ring R . First, consider the algebras $\hat{H}_{d,\mathbf{k}}(\zeta_e)$ and $\hat{H}_{d,K}(q_e)$ defined in the same way as in Section 2.11 with respect to the sets $\mathcal{F}(\zeta_e, Q) \subset \mathbf{k}$ and $\mathcal{F}(q_e, Q) \subset K$, where Q is as in Section 2.12. We can consider the R -algebra $\hat{H}_{d,R}(q_e)$ defined in a similar way with respect to the same set of idempotents as $\hat{H}_{d,\mathbf{k}}(q_e)$. Sometimes we will write $\hat{H}_{d,\mathbf{k}}^\nu(\zeta_e)$, $\hat{H}_{d,\mathbf{k}}^\nu(q_e)$ and $\hat{H}_{d,K}^\nu(q_e)$ to specify the parameter ν (the l -tuple Q in Section 2.12 depends on the l -tuple ν). Note that the algebra \hat{H}_d (over \mathbf{k} , R or K) admits the same block decomposition as the block decomposition for the cyclotomic Hecke algebra H_d^ν described above.

Now, we define the algebra $\widehat{SH}_{\alpha, \bar{k}}(\zeta_{e+1})$ that is a Hecke analogue of a localization of the balanced KLR algebra $S_{\alpha, \mathbf{k}}$. To do so, consider the idempotent $\mathbf{e} = \sum_{i \in \tilde{I}_{\text{ord}}^{\alpha}} e(i)$ in $\widehat{H}_{\alpha, \bar{k}}^{\bar{\nu}}(\zeta_{e+1})$. We set

$$\widehat{SH}_{\alpha, \bar{k}}(q_{e+1}) = \mathbf{e} \widehat{H}_{\alpha, \bar{R}}^{\bar{\nu}}(q_{e+1}) \mathbf{e} / \sum_{\mathbf{j} \in \tilde{I}_{\text{un}}^{\alpha}} \mathbf{e} \widehat{H}_{\alpha, \bar{R}}^{\bar{\nu}}(q_{e+1}) e(\mathbf{j}) \widehat{H}_{\alpha, \bar{R}}^{\bar{\nu}}(q_{e+1}) \mathbf{e}.$$

We define the K -algebra $\widehat{SH}_{\alpha, \bar{K}}(q_{e+1})$ as a similar quotient of $\mathbf{e} \widehat{H}_{\alpha, \bar{K}}^{\bar{\nu}}(q_{e+1}) \mathbf{e}$ by the two-sided ideal generated by $\{e(\mathbf{j}); \mathbf{j} \in \tilde{I}_{\text{un}}^{\alpha}\}$. Finally, we define the R -algebra $\widehat{SH}_{\alpha, \bar{R}}(q_{e+1})$ as the image in $\widehat{SH}_{\alpha, \bar{K}}(q_{e+1})$ of the following composition of homomorphisms

$$\mathbf{e} \widehat{H}_{\alpha, \bar{R}}^{\bar{\nu}}(q_{e+1}) \mathbf{e} \rightarrow \mathbf{e} \widehat{H}_{\alpha, \bar{K}}^{\bar{\nu}}(q_{e+1}) \mathbf{e} \rightarrow \widehat{SH}_{\alpha, \bar{K}}(q_{e+1}).$$

Remark 2.19. By Proposition 2.17 we have algebra isomorphisms

$$\begin{aligned} \widehat{R}_{\alpha, \mathbf{k}}(\Gamma) &\simeq \widehat{H}_{\alpha, \mathbf{k}}(\zeta_e), & \widehat{R}_{\alpha, K}(\tilde{\Gamma}) &\simeq \widehat{H}_{\alpha, K}(q_e), \\ \widehat{R}_{\alpha, \mathbf{k}}(\bar{\Gamma}) &\simeq \widehat{H}_{\alpha, \bar{k}}(\zeta_{e+1}), & \widehat{R}_{\alpha, K}(\tilde{\Gamma}) &\simeq \widehat{H}_{\alpha, \bar{K}}(q_{e+1}), \end{aligned}$$

from which we deduce that

$$\widehat{S}_{\alpha, \mathbf{k}}(\bar{\Gamma}) \simeq \widehat{SH}_{\alpha, \bar{k}}(\zeta_{e+1}), \quad \widehat{S}_{\alpha, K}(\tilde{\Gamma}) \simeq \widehat{SH}_{\alpha, \bar{K}}(q_{e+1}).$$

We may use these isomorphisms without mentioning them explicitly. Using the identifications above between KLR algebras and Hecke algebras, a localization of the isomorphism in Theorem 2.11 yields an isomorphism

$$\Phi_{\alpha, \mathbf{k}}: \widehat{H}_{\alpha, \mathbf{k}}(\zeta_e) \rightarrow \widehat{SH}_{\alpha, \mathbf{k}}(\zeta_{e+1}). \quad (18)$$

In the same way we also obtain an algebra isomorphism

$$\Phi_{\alpha, K}: \widehat{H}_{\alpha, K}(q_e) \rightarrow \widehat{SH}_{\tilde{\phi}(\alpha), K}(q_{e+1})$$

for each $\tilde{\alpha} \in Q_I^+$. Taking the sum over all $\tilde{\alpha} \in Q_I^+$ such that $\pi_e(\tilde{\alpha}) = \alpha$ yields a isomorphism

$$\Phi_{\alpha, K}: \widehat{H}_{\alpha, K}(q_e) \rightarrow \widehat{SH}_{\alpha, K}(q_{e+1}). \quad (19)$$

Lemma 2.20. *The homomorphism $\mathbf{e} \widehat{H}_{\alpha, \bar{R}}^{\bar{\nu}}(q_{e+1}) \mathbf{e} \rightarrow \mathbf{e} \widehat{H}_{\alpha, \bar{k}}^{\bar{\nu}}(\zeta_{e+1}) \mathbf{e}$ factors through a homomorphism $\widehat{SH}_{\alpha, \bar{R}}(q_{e+1}) \rightarrow \widehat{SH}_{\alpha, \bar{k}}(\zeta_{e+1})$.*

Proof. In Section 2.5 we constructed a faithful polynomial representation of $S_{\alpha, \mathbf{k}}$. Let us call it $\mathcal{Pol}_{\mathbf{k}}$. It is constructed as a quotient of the standard polynomial representation of $\mathbf{e} R_{\alpha, \mathbf{k}} \mathbf{e}$. After localization we get a faithful representation $\widehat{\mathcal{Pol}}_{\mathbf{k}}$ of $\widehat{S}_{\alpha, \mathbf{k}}$. Thus the kernel of the algebra homomorphism $\mathbf{e} \widehat{R}_{\alpha, \mathbf{k}} \mathbf{e} \rightarrow \widehat{S}_{\alpha, \mathbf{k}}$ is the annihilator of the representation $\widehat{\mathcal{Pol}}_{\mathbf{k}}$. We can transfer this to the Hecke side

and we obtain that the kernel of the algebra homomorphism $\mathbf{e}\widehat{H}_{\alpha,\mathbf{k}}^{\overline{\nu}}(\zeta_{e+1})\mathbf{e} \rightarrow \widehat{SH}_{\alpha,\mathbf{k}}(\zeta_{e+1})$ is the annihilator of the representation $\widehat{Pol}_{\mathbf{k}}$. Similarly, we can characterize the kernel of the algebra homomorphism $\mathbf{e}\widehat{H}_{\alpha,\overline{K}}^{\overline{\nu}}(q_{e+1})\mathbf{e} \rightarrow \widehat{SH}_{\alpha,\overline{K}}(q_{e+1})$ as the annihilator of a similar representation \widehat{Pol}_K .

The K -vector space \widehat{Pol}_K has an R -submodule \widehat{Pol}_R stable by the action of $\mathbf{e}\widehat{H}_{\alpha,\overline{R}}^{\overline{\nu}}(q_{e+1})\mathbf{e}$ such that $\mathbf{k} \otimes_R \widehat{Pol}_R = \widehat{Pol}_{\mathbf{k}}$ and it is compatible with the algebra homomorphism $\mathbf{e}\widehat{H}_{\alpha,\overline{R}}^{\overline{\nu}}(q_{e+1})\mathbf{e} \rightarrow \mathbf{e}\widehat{H}_{\alpha,\mathbf{k}}^{\overline{\nu}}(\zeta_{e+1})\mathbf{e}$. By definition of $\widehat{SH}_{\alpha,\overline{R}}(q_{e+1})$ and the discussion above, the kernel of the algebra homomorphism $\mathbf{e}\widehat{H}_{\alpha,\overline{R}}^{\overline{\nu}}(q_{e+1})\mathbf{e} \rightarrow \widehat{SH}_{\alpha,\overline{R}}(q_{e+1})$ is formed by the elements that act by zero on \widehat{Pol}_K . Thus each element of this kernel acts by zero on \widehat{Pol}_R . This implies, that an element of the kernel of $\mathbf{e}\widehat{H}_{\alpha,\overline{R}}^{\overline{\nu}}(q_{e+1})\mathbf{e} \rightarrow \widehat{SH}_{\alpha,\overline{R}}(q_{e+1})$ specializes to an element of the kernel of $\mathbf{e}\widehat{H}_{\alpha,\mathbf{k}}^{\overline{\nu}}(q_{e+1})\mathbf{e} \rightarrow \widehat{SH}_{\alpha,\mathbf{k}}(\zeta_{e+1})$. This proves the statement. \square

Lemma 2.21. *There is a unique algebra homomorphism $\Phi_{\alpha,R}: \widehat{H}_{\alpha,R}(q_e) \rightarrow \widehat{SH}_{\alpha,R}(q_{e+1})$ such that the following diagram is commutative*

$$\begin{array}{ccc} \widehat{H}_{\alpha,\mathbf{k}}(\zeta_e) & \xrightarrow{\Phi_{\alpha,\mathbf{k}}} & \widehat{SH}_{\alpha,\mathbf{k}}(\zeta_{e+1}) \\ \uparrow & & \uparrow \\ \widehat{H}_{\alpha,R}(q_e) & \xrightarrow{\Phi_{\alpha,R}} & \widehat{SH}_{\alpha,\overline{R}}(q_{e+1}) \\ \downarrow & & \downarrow \\ \widehat{H}_{\alpha,K}(q_e) & \xrightarrow{\Phi_{\alpha,K}} & \widehat{SH}_{\alpha,\overline{K}}(q_{e+1}). \end{array}$$

Proof. First we consider the algebras $H_{\alpha,\mathbf{k}}^{\text{loc}}(\zeta_e)$, $H_{\alpha,R}^{\text{loc}}(q_e)$ and $H_{\alpha,K}^{\text{loc}}(q_e)$ obtained from $\widehat{H}_{\alpha,\mathbf{k}}(\zeta_e)$, $\widehat{H}_{\alpha,R}(q_e)$ and $\widehat{H}_{\alpha,K}(q_e)$ by inverting

- $(X_r - X_t)$, $(\zeta_e X_r - X_t)$ with $r \neq t$,
- $(X_r - X_t)$, $(q_e X_r - X_t)$ with $r \neq t$,
- $(X_r - X_t)$, $(q_e X_r - X_t)$ with $r \neq t$

respectively. Let $SH_{\alpha,\mathbf{k}}^{\text{loc}}$ and $SH_{\alpha,\overline{K}}^{\text{loc}}$ be the localizations of $\widehat{SH}_{\alpha,\mathbf{k}}$ and $\widehat{SH}_{\alpha,\overline{K}}$ such that the isomorphisms $\Phi_{\alpha,\mathbf{k}}$ and $\Phi_{\alpha,K}$ above induce isomorphisms

$$\Phi_{\alpha,\mathbf{k}}: H_{\alpha,\mathbf{k}}^{\text{loc}}(\zeta_e) \rightarrow SH_{\alpha,\mathbf{k}}^{\text{loc}}(\zeta_{e+1})$$

$$\Phi_{\alpha,K}: H_{\alpha,K}^{\text{loc}}(q_e) \rightarrow SH_{\alpha,\overline{K}}^{\text{loc}}(q_{e+1}).$$

Let $SH_{\alpha, \overline{R}}^{\text{loc}}$ be the image in $SH_{\alpha, \overline{K}}^{\text{loc}}$ of the following composition of homomorphisms

$$\mathbf{e}H_{\alpha, \overline{R}}^{\text{loc}}(q_{e+1})\mathbf{e} \rightarrow \mathbf{e}H_{\alpha, \overline{K}}^{\text{loc}}(q_{e+1})\mathbf{e} \rightarrow SH_{\alpha, \overline{K}}^{\text{loc}}(q_{e+1}).$$

Next, we want to prove that there exists an algebra homomorphism $\Phi_{\alpha, R}: H_{\alpha, R}^{\text{loc}}(q_e) \rightarrow SH_{\alpha, \overline{R}}^{\text{loc}}(q_{e+1})$ such that the following diagram is commutative:

$$\begin{array}{ccc} H_{\alpha, \mathbf{k}}^{\text{loc}}(\zeta_e) & \xrightarrow{\Phi_{\alpha, \mathbf{k}}} & SH_{\alpha, \overline{\mathbf{k}}}^{\text{loc}}(\zeta_{e+1}) \\ \uparrow & & \uparrow \\ H_{\alpha, R}^{\text{loc}}(q_e) & \xrightarrow{\Phi_{\alpha, R}} & SH_{\alpha, \overline{R}}^{\text{loc}}(q_{e+1}) \\ \downarrow & & \downarrow \\ H_{\alpha, K}^{\text{loc}}(q_e) & \xrightarrow{\Phi_{\alpha, K}} & SH_{\alpha, \overline{K}}^{\text{loc}}(q_{e+1}). \end{array} \quad (20)$$

We just need to check that the map $\Phi_{\alpha, K}$ takes an element of $H_{\alpha, R}^{\text{loc}}(q_e)$ to an element of $SH_{\alpha, \overline{R}}^{\text{loc}}(q_{e+1})$ and that it specializes to the map $\Phi_{\alpha, \mathbf{k}}: H_{\alpha, \mathbf{k}}^{\text{loc}}(\zeta_e) \rightarrow SH_{\alpha, \overline{\mathbf{k}}}^{\text{loc}}(\zeta_{e+1})$. We will check this on the generators $e(\mathbf{i})$, $X_r e(\mathbf{i})$ and $\Psi_r e(\mathbf{i})$ of $H_{\alpha, R}^{\text{loc}}(q_e)$.

This is obvious for the idempotents $e(\mathbf{i})$.

Let us check this for $X_r e(\mathbf{i})$. Assume that $\mathbf{i} \in I^\alpha$ and $\mathbf{j} \in \tilde{I}^{|\alpha|}$ are such that we have $\pi_e(\mathbf{j}) = \mathbf{i}$. Write $\mathbf{i}' = \phi(\mathbf{i})$ and $\mathbf{j}' = \tilde{\phi}(\mathbf{j})$. Set $r' = r_{\mathbf{j}}' = r_{\mathbf{i}}'$, see the notation in Section 2.6. By Theorem 2.11 and Proposition 2.16 we have

$$\Phi_{\alpha, K}(X_r e(\mathbf{j})) = \overline{p}_{j_{r'}}^{-1} p_{j_r} X_{r'} e(\mathbf{j}').$$

Since, $\overline{p}_{j_{r'}}^{-1} p_{j_r}$ depends only on \mathbf{i} and r and $e(\mathbf{i}) = \sum_{\pi_e(\mathbf{j})=\mathbf{i}} e(\mathbf{j})$, we deduce that

$$\Phi_{\alpha, K}(X_r e(\mathbf{i})) = \overline{p}_{j_{r'}}^{-1} p_{j_r} X_{r'} e(\mathbf{i}').$$

Thus the element $\Phi_{\alpha, K}(X_r e(\mathbf{i}))$ is in $SH_{\alpha, \overline{R}}^{\text{loc}}$ and its image in $SH_{\alpha, \overline{\mathbf{k}}}^{\text{loc}}$ is $\overline{p}_{i_{r'}}^{-1} p_{i_r} X_{r'} e(\mathbf{i}') = \Phi_{\alpha, \mathbf{k}}(X_r e(\mathbf{i}'))$.

Next, we consider the generators $\Psi_r e(\mathbf{i})$. We must prove that for each \mathbf{j} such that $\pi_e(\mathbf{j}) = \mathbf{i}$ and for each r we have

- $\Phi_{\alpha, K}(\Psi_r e(\mathbf{j})) = \Xi e(\mathbf{j}')$, for some element $\Xi \in H_{\alpha, R}^{\text{loc}}(q_e)$ that depends only on r and \mathbf{i} ,
- the image of $\Xi e(\mathbf{i}')$ in $SH_{\alpha, \overline{\mathbf{k}}}^{\text{loc}}(q_{e+1})$ under the specialization $R \rightarrow \mathbf{k}$ is $\Phi_{\alpha, \mathbf{k}}(\Psi_r e(\mathbf{i}'))$.

This follows from Lemma 2.12.

Now we obtain the diagram from the claim of Lemma 2.21 as the restriction of the diagram (20). □

3 The category \mathcal{O}

3.1 Affine Lie algebras

Fix positive integers N, l and e such that $e > 1$. Let R be a deformation ring, see Section 2.9. Set

$$\mathfrak{g}_R = \mathfrak{gl}_N(R), \quad \widehat{\mathfrak{g}}_R = \widehat{\mathfrak{gl}}_N(R) = \mathfrak{gl}_N(R)[t, t^{-1}] \oplus R\mathbf{1} \oplus R\partial.$$

For $i, j \in [1, N]$ let $e_{i,j} \in \mathfrak{g}_R$ denote the matrix unit. Let $\mathfrak{h}_R \subset \mathfrak{g}_R$ be the Cartan subalgebra generated by the $e_{i,i}$'s, and $\epsilon_1, \dots, \epsilon_N$ be the basis of \mathfrak{h}_R^* dual to $e_{1,1}, \dots, e_{N,N}$. Let $P = \mathbb{Z}\epsilon_1 \oplus \dots \oplus \mathbb{Z}\epsilon_N$ be the weight lattice of \mathfrak{g}_R . We identify P with \mathbb{Z}^N . Let $\Pi, \widehat{\Pi} \subset \mathfrak{h}_R^*$ be the sets of simple roots of \mathfrak{g}_R and $\widehat{\mathfrak{g}}_R$. Let $W = \mathfrak{S}_N$ be the Weyl group of \mathfrak{g}_R and $\widehat{W} = W \ltimes \mathbb{Z}\Pi, \widehat{W} = W \ltimes P$ be the affine and the extended affine Weyl groups.

Recall that $I \in \mathbb{Z}/e\mathbb{Z}$. We still consider the quiver $\Gamma = (I, H)$ as in Section 2.8. In particular, we have $X_I = X_e, P_I = P_e$, see Section 2.1 for the notation. Then we define the element $\mathbf{wt}_e(\lambda) \in X_I$ given by

$$\mathbf{wt}_e(\lambda) = \sum_{s=1}^N \varepsilon_{\lambda_s},$$

where we write ε_{λ_r} for $\varepsilon_{(\lambda_r \bmod e)}$. We will abbreviate

$$P[\mu] = \{\lambda \in P; \mathbf{wt}_e(\lambda) = \mu\}. \quad (21)$$

Similarly, we consider the weight

$$\mathbf{wt}_e^\delta(\lambda) = \sum_{r=1}^N \varepsilon_{\lambda_r} + \left(\sum_{r=1}^N \lambda_r\right)\delta \in X_e^\delta.$$

Finally, let $X_I[N] \subset X_I$ be the subset given by

$$X_I[N] = \left\{ \mu = \sum_{r=1}^e \mu_r \varepsilon_r \in X_I; \mu_r \geq 0, \sum_{r=1}^e \mu_r = N \right\}.$$

We may identify μ with the tuple (μ_1, \dots, μ_e) if no confusion is possible.

Now, consider the Cartan subalgebra $\widehat{\mathfrak{h}}_R = \mathfrak{h}_R \oplus R\mathbf{1} \oplus R\partial$ of $\widehat{\mathfrak{g}}_R$. Let Λ_0 and δ be the elements of $\widehat{\mathfrak{h}}_R^*$ defined by

$$\delta(\partial) = \Lambda_0(\mathbf{1}) = 1, \quad \delta(\mathfrak{h}_R \oplus R\mathbf{1}) = \Lambda_0(\mathfrak{h}_R \oplus R\partial) = 0.$$

Let $(\bullet, \bullet): \widehat{\mathfrak{h}}_R^* \times \widehat{\mathfrak{h}}_R^* \rightarrow R$ be the bilinear form such that

$$\lambda(\alpha_r^\vee) = (\lambda, \alpha_r), \quad \lambda(\partial) = (\lambda, \Lambda_0), \quad \forall \lambda \in \widehat{\mathfrak{h}}_R^*.$$

Set $P_R = P \otimes_{\mathbb{Z}} R$. Given a composition $\nu = (\nu_1, \dots, \nu_l)$ of N , we define

$$\begin{aligned} \rho &= (0, -1, \dots, -N+1), \\ \rho_\nu &= (\nu_1, \nu_1 - 1, \dots, 1, \nu_2, \dots, 1, \dots, \nu_l, \dots, 1), \\ \tau &= (\tau_1^{\nu_1}, \dots, \tau_l^{\nu_l}), \end{aligned}$$

where $\tau_r^{\nu_r}$ means ν_r copies of τ_r . Set also

$$\widehat{\rho} = \rho + N\Lambda_0, \quad \widetilde{\lambda} = \lambda + \tau + z_\lambda \delta - (N + \kappa)\Lambda_0, \quad (22)$$

where $z_\lambda = (\lambda, 2\rho + \lambda)/2\kappa$. Denote by $\widehat{\mathfrak{p}}_{R,\nu}$ the parabolic subalgebra of $\widehat{\mathfrak{g}}_R$ of parabolic type ν . For a ν -dominant weight $\lambda \in P$ let $\Delta(\lambda)_R$ be the parabolic Verma module with highest weight $\widetilde{\lambda}$ and $\Delta_R^\lambda = \Delta(\lambda - \rho)_R$. We will also skip the subscript R when $R = \mathbb{C}$.

3.2 Extended affine Weyl groups

Assume that $R = \mathbb{C}$. In this section we discuss some combinatorial aspects of the \widehat{W} -action on $\widehat{\mathfrak{h}}^*$.

The group \widehat{W} is generated by $\{\pi, s_i; i \in \mathbb{Z}/N\mathbb{Z}\}$ modulo the relations

$$\begin{aligned} s_i^2 &= 1, \\ s_i s_j &= s_j s_i \quad \forall i \neq j \pm 1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ \pi s_{i+1} &= s_i \pi. \end{aligned}$$

Let \widetilde{W} be the subgroup of \widehat{W} generated by $\{s_i; i \in \mathbb{Z}/N\mathbb{Z}\}$. The group \widehat{W} acts on P in the following way:

- s_r switches of the r th and $(r+1)$ th components of λ if $r \neq 0$,
- $s_0(\lambda_1, \dots, \lambda_N) = (\lambda_N - e, \lambda_2, \dots, \lambda_{N-1}, \lambda_1 + e)$,
- $\pi(\lambda_1, \dots, \lambda_N) = (\lambda_2, \dots, \lambda_N, \lambda_1 + e)$.

We will call this action of \widehat{W} on P the negative e -action. We will always consider only negative actions of \widehat{W} on P up to Section 5.7. So we can skip the word "negative". We may write $P^{(e)} = P$ to stress that we consider the e -action of \widehat{W} on P . The map

$$P^{(e)} \rightarrow \widehat{\mathfrak{h}}^*, \quad \lambda \mapsto \widetilde{\lambda - \rho} + \widehat{\rho}$$

is \widehat{W} -invariant. This means that the weights $\lambda_1, \lambda_2 \in P$ are in the same \widehat{W} -orbit if and only if the highest weights of the Verma modules Δ^{λ_1} and Δ^{λ_2} are linked with respect to the Weyl group \widehat{W} , see [21, Sec. 3.2] and [8, Sec. 2.3] for more details about linkage. Note that $P = \coprod_{\mu \in X_I[N]} P[\mu]$ is the decomposition of P

into \widehat{W} -orbits with respect to the e -action. An element $\lambda \in P$ is e -anti-dominant if $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N \leq \lambda_1 + e$.

Recall the map $\Upsilon: \mathbb{Z} \rightarrow \mathbb{Z}$ from (10). Applying Υ coordinate by coordinate to the elements of P we get a map $\Upsilon: P^{(e)} \rightarrow P^{(e+1)}$.

Lemma 3.1. *The map $\Upsilon: P^{(e)} \rightarrow P^{(e+1)}$ is \widehat{W} -invariant and takes e -anti-dominant weights to $(e+1)$ -anti-dominant weights. \square*

3.3 The standard representation of $\widetilde{\mathfrak{sl}}_e$

Let e_i, f_i, h_i the generators of the complex Lie algebra $\widetilde{\mathfrak{sl}}_e = \mathfrak{sl}_e \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{1}$. Let V_e be a \mathbb{C} -vector spaces with canonical basis $\{v_1, \dots, v_e\}$ and set $U_e = V_e \otimes \mathbb{C}[z, z^{-1}]$. The vector space U_e has a basis $\{u_r; r \in \mathbb{Z}\}$ where $u_{a+eb} = v_a \otimes z^{-b}$ for $a \in [1, e], b \in \mathbb{Z}$. It has a structure of an $\widetilde{\mathfrak{sl}}_e$ -module such that

$$f_i(u_r) = \delta_{i \equiv r} u_{r+1}, \quad e_i(u_r) = \delta_{i \equiv r-1} u_{r-1}.$$

Let $\{v'_1, \dots, v'_{e+1}\}, \{u'_r; r \in \mathbb{Z}\}$ denote the bases of V_{e+1} and U_{e+1} .

Fix an integer $0 \leq k < e$. Consider the following inclusion of vector spaces

$$V_e \subset V_{e+1}, \quad v_r \mapsto \begin{cases} v'_r & \text{if } r \leq k, \\ v'_{r+1} & \text{if } r > k. \end{cases}$$

It yields an inclusion $\mathfrak{sl}_e \subset \mathfrak{sl}_{e+1}$ such that

$$\begin{aligned} e_r &\mapsto \begin{cases} e_r & \text{if } r \in [1, k-1], \\ [e_k, e_{k+1}] & \text{if } r = k, \\ e_{r+1} & \text{if } r \in [k+1, e-1], \end{cases} \\ f_r &\mapsto \begin{cases} f_r & \text{if } r \in [1, k-1], \\ [f_{k+1}, f_k] & \text{if } r = k, \\ f_{r+1} & \text{if } r \in [k+1, e-1], \end{cases} \\ h_r &\mapsto \begin{cases} h_r & \text{if } r \in [1, k-1], \\ h_k + h_{k+1} & \text{if } r = k, \\ h_{r+1} & \text{if } r \in [k+1, e-1]. \end{cases} \end{aligned}$$

This inclusion lifts uniquely to an inclusion $\widetilde{\mathfrak{sl}}_e \subset \widetilde{\mathfrak{sl}}_{e+1}$ such that

$$\begin{aligned} e_0 &\mapsto \begin{cases} e_0 & \text{if } k \neq 0, \\ [e_0, e_1] & \text{else,} \end{cases} \\ f_0 &\mapsto \begin{cases} f_0 & \text{if } k \neq 0, \\ [f_1, f_0] & \text{else,} \end{cases} \\ h_0 &\mapsto \begin{cases} h_0 & \text{if } k \neq 0, \\ h_0 + h_1 & \text{else.} \end{cases} \end{aligned}$$

Consider the inclusion $U_e \subset U_{e+1}$ such that $u_r \mapsto u'_{\Upsilon(r)}$.

Lemma 3.2. *The embeddings $V_e \subset V_{e+1}$ and $U_e \subset U_{e+1}$ are compatible with the actions of $\mathfrak{sl}_e \subset \mathfrak{sl}_{e+1}$ and $\widetilde{\mathfrak{sl}}_e \subset \widetilde{\mathfrak{sl}}_{e+1}$ respectively. \square*

3.4 Categorical representations

Let R be a \mathbb{C} -algebra. Fix an invertible element $q \in R$, $q \neq 1$. Let \mathcal{C} be an exact R -linear category.

Definition 3.3. A *representation datum* in \mathcal{C} is a tuple (E, F, X, T) where (E, F) is a pair of exact functors $\mathcal{C} \rightarrow \mathcal{C}$ and $X \in \text{End}(F)^{\text{op}}$, $T \in \text{End}(F^2)^{\text{op}}$ are endomorphisms of functors such that for each $d \in \mathbb{N}$, there is an R -algebra homomorphism $\psi_d: H_{d,R}(q) \rightarrow \text{End}(F^d)^{\text{op}}$ given by

$$\begin{aligned} X_r &\mapsto F^{d-r} X F^{r-1} & \forall r \in [1, d], \\ T_r &\mapsto F^{d-r-1} T F^{r-1} & \forall r \in [1, d-1]. \end{aligned}$$

Now, assume that $R = \mathbf{k}$ is a field. Assume that \mathcal{C} is a Hom-finite abelian category. Let \mathcal{F} be a subset of \mathbf{k} such that $q^{\mathbb{Z}}\mathcal{F}/q^{\mathbb{Z}}$ is finite. We view \mathcal{F} as the vertex set of a quiver with an arrow $i \rightarrow j$ if and only if $j = qi$.

Remark 3.4. Assume that we have a representation datum in a \mathbf{k} -linear category \mathcal{C} such that the functors E and F are biadjoint. Then by adjointness we have an algebra isomorphism $\text{End}(E^d) \simeq \text{End}(F^d)^{\text{op}}$. In particular we get an algebra homomorphism $H_{d,\mathbf{k}} \rightarrow \text{End}(E^d)$.

Definition 3.5. An $\mathfrak{sl}_{\mathcal{F}}$ -categorical representation in \mathcal{C} is the datum of a representation datum (E, F, X, T) and a decomposition $\mathcal{C} = \bigoplus_{\mu \in X_{\mathcal{F}}} \mathcal{C}_{\mu}$ satisfying the conditions (a) and (b) below. For $i \in \mathcal{F}$ let E_i, F_i be endofunctors of \mathcal{C} such that for each $M \in \mathcal{C}$ the objects $E_i(M), F_i(M)$ are the generalized i -eigenspaces of X acting on $E(M)$ and $F(M)$ respectively, see also Remark 3.4. We assume that

- (a) $F = \bigoplus_{i \in \mathcal{F}} F_i$ and $E = \bigoplus_{i \in \mathcal{F}} E_i$,
- (b) $E_i(\mathcal{C}_{\mu}) \subset \mathcal{C}_{\mu+\alpha_i}$ and $F_i(\mathcal{C}_{\mu}) \subset \mathcal{C}_{\mu-\alpha_i}$.

Remark 3.6. In this case the homomorphism $\psi_d: H_{d,\mathbf{k}} \rightarrow \text{End}(F^d)^{\text{op}}$ extends to a homomorphism $\hat{H}_{d,\mathbf{k}} \rightarrow \text{End}(F^d)^{\text{op}}$, where $\hat{H}_{d,\mathbf{k}}$ is as in Section 2.11.

There is an alternative definition of a categorical representation, where the affine Hecke algebra $H_{d,\mathbf{k}}(q)$ is replaced by a KLR algebra. In this section we allow $\Gamma = (I, H)$ be an arbitrary quiver without 1-loops.

Definition 3.7. A \mathfrak{g}_I -categorical representation (E, F, x, τ) in \mathcal{C} is the following data:

- (1) a decomposition $\mathcal{C} = \bigoplus_{\mu \in X_I} \mathcal{C}_{\mu}$,
- (2) a pair of biadjoint exact endofunctors (E, F) of \mathcal{C} ,
- (3) morphisms of functors $x: F \rightarrow F$, $\tau: F^2 \rightarrow F^2$,
- (4) decompositions $E = \bigoplus_{i \in I} E_i$, $F = \bigoplus_{i \in I} F_i$,

satisfying the following conditions.

- (a) We have $E_i(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu+\alpha_i}$, $F_i(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu-\alpha_i}$.
- (b) For each $d \in \mathbb{N}$ there is an algebra homomorphism $\psi_d: R_{d,\mathbf{k}} \rightarrow \text{End}(F^d)^{\text{op}}$ such that $\psi_d(e(\mathbf{i}))$ is the projector to $F_{i_d} \cdots F_{i_1}$, where $\mathbf{i} = (i_1, \dots, i_d)$ and

$$\psi_d(x_r) = F^{d-r} x F^{r-1}, \quad \psi_d(\tau_r) = F^{d-r-1} \tau F^{r-1}.$$

- (c) For each $M \in \mathcal{C}$ the endomorphism of $F(M)$ induced by x is nilpotent.

Assume that there is an isomorphism of quivers $I \simeq \mathcal{F}$. Then Definitions 3.5, 3.7 are equivalent by Proposition 2.17.

3.5 From $\widetilde{\mathfrak{sl}}_{e+1}$ -categorical representations to $\widetilde{\mathfrak{sl}}_e$ -categorical representations

Recall that we fix $\Gamma = (I, H)$ and $\overline{\Gamma} = (\overline{I}, \overline{H})$ as in Section 2.8. In particular, we have $X_I = X_e$, $X_{\overline{I}} = X_{e+1}$.

Assume that $R = \mathbf{k}$ is a field. Let $\overline{\mathcal{C}}$ be a Hom-finite abelian \mathbf{k} -linear category. Let

$$\overline{E} = \overline{E}_0 \oplus \overline{E}_1 \oplus \cdots \oplus \overline{E}_e, \quad \overline{F} = \overline{F}_0 \oplus \overline{F}_1 \oplus \cdots \oplus \overline{F}_e$$

be endofunctors defining a categorical $\widetilde{\mathfrak{sl}}_{e+1}$ -representation in $\overline{\mathcal{C}}$. Let $\overline{\psi}_d: R_{d,\mathbf{k}} \rightarrow \text{End}(\overline{F}^d)^{\text{op}}$ be the corresponding algebra homomorphism. We set $\overline{F}_{\mathbf{i}} = \overline{F}_{i_d} \cdots \overline{F}_{i_1}$ for any tuple $\mathbf{i} = (i_1, \dots, i_d) \in \overline{I}^d$ and $\overline{F}_{\overline{\alpha}} = \bigoplus_{\mathbf{i} \in \overline{I}^{\overline{\alpha}}} \overline{F}_{\mathbf{i}}$ for any element $\overline{\alpha} \in Q_{\overline{I}}^+$. If $|\overline{\alpha}| = d$ let $\overline{\psi}_{\overline{\alpha}}: R_{\overline{\alpha},\mathbf{k}} \rightarrow \text{End}(\overline{F}_{\overline{\alpha}})^{\text{op}}$ be the $\overline{\alpha}$ -component of $\overline{\psi}_d$.

Now, recall the notation X_I^+ from (2). Assume that

$$\overline{\mathcal{C}}_\mu = 0, \quad \forall \mu \in X_{\overline{I}} \setminus X_I^+. \quad (23)$$

For $\mu \in X_I^+$ set $\mathcal{C}_\mu = \overline{\mathcal{C}}_{\phi(\mu)}$, where the map ϕ is as in (4). Let $\mathcal{C} = \bigoplus_{\mu \in X_I^+} \mathcal{C}_\mu$.

Remark 3.8. (a) \mathcal{C} is stable by $\overline{F}_i, \overline{E}_i$ for each $i \neq k, k+1$,

(b) \mathcal{C} is stable by $\overline{F}_{k+1}\overline{F}_k, \overline{E}_k\overline{E}_{k+1}$,

(c) $\overline{F}_{i_d}\overline{F}_{i_{d-1}} \cdots \overline{F}_{i_1}(M) = 0$ for each $M \in \mathcal{C}$ whenever the sequence (i_1, \dots, i_d) is unordered (see Sections 2.4 and 2.8).

Consider the following endofunctors of \mathcal{C} :

$$E_i = \begin{cases} \overline{E}_i|_{\mathcal{C}} & \text{if } 0 \leq i < k, \\ \overline{E}_k\overline{E}_{k+1}|_{\mathcal{C}} & \text{if } i = k, \\ \overline{E}_{i+1}|_{\mathcal{C}} & \text{if } k < i < e, \end{cases}$$

$$F_i = \begin{cases} \overline{F}_i|_{\mathcal{C}} & \text{if } 0 \leq i < k, \\ \overline{F}_{k+1}\overline{F}_k|_{\mathcal{C}} & \text{if } i = k, \\ \overline{F}_{i+1}|_{\mathcal{C}} & \text{if } k < i < e. \end{cases}$$

Similarly to the notations above we set $F_{\mathbf{i}} = F_{i_d} \cdots F_{i_1}$ for any tuple $\mathbf{i} = (i_1, \dots, i_d) \in I^d$ and $F_\alpha = \bigoplus_{\mathbf{i} \in I^\alpha} F_{\mathbf{i}}$ for any element $\alpha \in Q_I^+$. Note that we have $F_{\mathbf{i}} = \overline{F}_{\phi(\mathbf{i})}|_{\mathcal{C}}$ for each $\mathbf{i} \in I^\alpha$.

Let $\alpha \in \mathbb{N}I$ and $\overline{\alpha} = \phi(\alpha)$. Note that we have

$$F_\alpha = \bigoplus_{\mathbf{i} \in \overline{I}_{\text{ord}}^\alpha} \overline{F}_{\mathbf{i}}|_{\mathcal{C}}.$$

The homomorphism $\overline{\psi}_{\overline{\alpha}}$ yields a homomorphism $\mathbf{e}R_{\overline{\alpha}, \mathbf{k}}\mathbf{e} \rightarrow \text{End}(F_\alpha)^{\text{op}}$, where $\mathbf{e} = \sum_{\mathbf{i} \in \overline{I}_{\text{ord}}^\alpha} e(\mathbf{i})$. By (c), the homomorphism $\mathbf{e}R_{\overline{\alpha}, \mathbf{k}}\mathbf{e} \rightarrow \text{End}(F_\alpha)^{\text{op}}$ factors through a homomorphism $S_{\overline{\alpha}, \mathbf{k}} \rightarrow \text{End}(F_\alpha)^{\text{op}}$. Let us call it $\overline{\psi}'_{\overline{\alpha}}$. Then we can define an algebra homomorphism $\psi_\alpha: R_{\alpha, \mathbf{k}} \rightarrow \text{End}(F_\alpha)^{\text{op}}$ by setting $\psi_\alpha = \overline{\psi}'_{\overline{\alpha}} \circ \Phi_{\alpha, \mathbf{k}}$.

Now, Theorem 2.11 implies the following result.

Lemma 3.9. *For each category $\overline{\mathcal{C}}$ as above that satisfies (23), we have a categorical representation of $\widetilde{\mathfrak{sl}}_e$ in the subcategory \mathcal{C} of $\overline{\mathcal{C}}$ given by functors F_i , E_i and the algebra homomorphisms $\psi_\alpha: R_{\alpha, \mathbf{k}} \rightarrow \text{End}(F_\alpha)^{\text{op}}$. \square*

3.6 The category \mathcal{O}

Let R be a deformation ring. Fix a composition $\nu = (\nu_1, \dots, \nu_l)$ of N . First we define an R -deformed version of the parabolic category \mathcal{O} for $\widehat{\mathfrak{gl}}_N$. Recall that we identify the weight lattice P with \mathbb{Z}^N . We say that $\lambda \in P$ is ν -dominant if $\lambda_r > \lambda_{r+1}$ for each $r \in [1, N-1] \setminus \{\nu_1, \nu_1 + \nu_2, \dots, \nu_1 + \dots + \nu_l\}$. Let P^ν be the set of ν -dominant weights of P . Set also $P^\nu[\mu] = P^\nu \cap P[\mu]$, where $P[\mu]$ is as in (21). Let O_R^ν be the R -linear abelian category of finitely generated $\widehat{\mathfrak{g}}_R$ -modules M which are weight $\widehat{\mathfrak{h}}_R$ -modules, and such that the $\widehat{\mathfrak{p}}_{R, \nu}$ -action on M is locally finite over R , and the highest weight of any subquotient of M is of the form $\widetilde{\lambda}$ with $\lambda \in P^\nu$, where $\widetilde{\lambda}$ is defined in (22). Let $O_{\mu, R}^\nu$ be the Serre subcategory of O_R^ν generated by the modules Δ_R^λ for all $\lambda \in P^\nu[\mu]$. Let $O_{\mu, R}^{\nu, \Delta} \subset O_{\mu, R}^\nu$ be the full subcategory of Δ -filtered modules.

We will omit the upper index ν if $\nu = (1, 1, \dots, 1)$. Assume $\lambda \in P$. In the case if $R = \mathbf{k}$ is a field we denote by $L(\lambda)_{\mathbf{k}}$ the simple quotient of $\Delta(\lambda)_{\mathbf{k}}$. In the case if R is local with residue fields \mathbf{k} , the simple module $L(\lambda)_{\mathbf{k}} \in O_{\mathbf{k}}$ has a simple lift $L(\lambda)_R \in O_R$ such that $L(\lambda)_{\mathbf{k}} = \mathbf{k} \otimes_R L(\lambda)_R$ (see [7, Sec. 2.2]). Set also $L_R^\lambda = L(\lambda - \rho)_R$.

Let U_e, V_e be as in Section 3.3. Set $\wedge^\nu U_e = \wedge^{\nu_1} U_e \otimes \cdots \otimes \wedge^{\nu_l} U_e$. For each $\lambda \in P^\nu$ define the following element in $\wedge^\nu U_e$:

$$\wedge_\lambda^\nu = (u_{\lambda_1} \wedge \cdots \wedge u_{\lambda_{\nu_1}}) \otimes \cdots \otimes (u_{\lambda_1 + \dots + \lambda_{l-1} + 1} \wedge \cdots \wedge u_{\lambda_{\nu_1} + \dots + \lambda_{\nu_l}}).$$

The obvious $\widetilde{\mathfrak{sl}}_e$ -action on U_e yields an $\widetilde{\mathfrak{sl}}_e$ -action on $\wedge^\nu U_e$. We identify the abelian group $X_e/\mathbb{Z}(\varepsilon_1 + \dots + \varepsilon_e)$ with the weight lattice of \mathfrak{sl}_e . In particular

each element $\mu \in X_e$ yields a weight of \mathfrak{sl}_e . For each $\mu \in X_e[N]$ let $(\wedge^\nu U_e)_\mu$ be the weight space in $\wedge^\nu U_e$ corresponding to μ .

Set $O_{-e,R}^\nu = \bigoplus_{\mu \in X_e[N]} O_{\mu,R}^\nu$ and similarly for $O_{-e,R}^{\nu,\Delta}$. Now we define a representation datum in the category $O_{-e,R}^{\nu,\Delta}$. See [19, Sec. 5.4] for more details. From now on, we assume that R is either a local analytic deformation ring in general position of dimension ≤ 2 or a field. Let \mathbf{k} and K be its residue field and field of fractions respectively. For an exact category \mathcal{C} denote by $[\mathcal{C}]$ its complexified Grothendieck group. The following proposition holds, see [19].

Proposition 3.10. *There is a pair of exact endofunctors E, F of $O_{-e,R}^{\nu,\Delta}$ such that the following properties hold.*

- (a) *The functors E, F commute with the base changes $K \otimes_R \bullet, \mathbf{k} \otimes_R \bullet$.*
- (b) *$(O_{-e,R}^{\nu,\Delta}, E, F)$ admits a representation datum structure.*
- (c) *The pair of functors (E, F) is biadjoint. It extends to a pair of biadjoint functors $O_{-e,R}^\nu \rightarrow O_{-e,R}^\nu$ if R is a field.*
- (d) *There are decompositions $E = \bigoplus_{i \in I} E_i, F = \bigoplus_{i \in I} F_i$ such that*

$$E_i(O_{\mu,R}^{\nu,\Delta}) \subset O_{\mu+\alpha_i,R}^{\nu,\Delta}, \quad F_i(O_{\mu,R}^{\nu,\Delta}) \subset O_{\mu-\alpha_i,R}^{\nu,\Delta}.$$

- (e) *There is a vector space isomorphism $[O_{\mu,R}^{\nu,\Delta}] \simeq (\wedge^\nu U)_\mu$ such that the functors F_i, E_i act on $[O_{-e,R}^{\nu,\Delta}] = \bigoplus_{\mu \in X_e[N]} [O_{\mu,R}^{\nu,\Delta}]$ as the standard generators e_i, f_i of $\widetilde{\mathfrak{sl}}_e$.*
- (f) *If $R = \mathbf{k}$ with the trivial deformation ring structure, then E_i, F_i yield a categorical representation of $\widetilde{\mathfrak{sl}}_e$ in $O_{-e,\mathbf{k}}^\nu$. \square*

Fix $k \in [0, e-1]$. Recall the map $\Upsilon: P \rightarrow P$ from Section 3.2 and the map $\phi: X_I \rightarrow X_{\overline{I}}$ from (4). Set $\mu' = \mu - \alpha_k$ and $\overline{\mu} = \phi(\mu)$. Set, $\overline{\mu}^0 = \overline{\mu} - \overline{\alpha}_k$ and $\overline{\mu}' = \overline{\mu} - \overline{\alpha}_k - \overline{\alpha}_{k+1}$. Note that $\Upsilon(P[\mu]) \subset P[\overline{\mu}]$. For $k \neq 0$ we have

$$\begin{aligned} \mu &= (\mu_1, \dots, \mu_k, & \mu_{k+1}, & \dots, \mu_e), \\ \mu' &= (\mu_1, \dots, \mu_k - 1, & \mu_{k+1} + 1, & \dots, \mu_e), \\ \overline{\mu} &= (\mu_1, \dots, \mu_k, & 0, & \mu_{k+1}, & \dots, \mu_e), \\ \overline{\mu}^0 &= (\mu_1, \dots, \mu_k - 1, & 1, & \mu_{k+1}, & \dots, \mu_e), \\ \overline{\mu}' &= (\mu_1, \dots, \mu_k - 1, & 0, & \mu_{k+1} + 1, & \dots, \mu_e). \end{aligned}$$

For an e -tuple $\mathbf{a} = (a_1, \dots, a_e)$ of non-negative integers we set $1_{\mathbf{a}} = (1^{a_1}, \dots, e^{a_e})$. Note that we have

$$\Upsilon(1_\mu) = 1_{\overline{\mu}}, \quad \Upsilon(1_{\mu'}) = 1_{\overline{\mu}'}. \quad (24)$$

Remark 3.11. The set $P[\mathbf{a}]$ is a \widehat{W} -orbit in $P^{(e)}$. It is a union of \widetilde{W} -orbits and each of them contains a unique e -anti-dominant weight. By definition, the weight $1_{\mathbf{a}} \in P[\mathbf{a}]$ is e -anti-dominant. However, there is no canonical way to choose $1_{\mathbf{a}}$. In the case $k = 0$ we need to change our convention and to set $1_{\mathbf{a}} = (0^{a_e}, 1^{a_e}, \dots, (e-1)^{a_{e-1}})$. This change is necessary to have (24).

First, assume that $l = N$ and $\nu = (1, 1, \dots, 1)$.

Lemma 3.12. *There is an equivalence of categories $\theta_\mu^\mu: O_{\mu,R} \rightarrow O_{\bar{\mu},\bar{R}}$ such that $\theta_\mu^\mu(\Delta_R^\lambda) \simeq \Delta_{\bar{R}}^{\Upsilon(\lambda)}$. It restricts to an equivalence of categories $\theta_\mu^\mu: O_{\mu,R}^\Delta \rightarrow O_{\bar{\mu},\bar{R}}^\Delta$.*

Proof. For each $n \in \mathbb{Z}$ the weight $\pi^n(1_\mu)$ is e -anti-dominant. Let $\mathcal{O}_{\pi^n(1_\mu),R} \subset O_{\mu,R}$ be the Serre subcategory generated by the Verma modules of the form $\Delta_R^{w\pi^n(1_\mu)}$ with $w \in \widetilde{W}$. We have

$$O_{\mu,R} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_{\pi^n(1_\mu),R}. \quad (25)$$

The weights $\pi^n(1_\mu) \in P^{(e)}$ and $\pi^n(1_{\bar{\mu}}) \in P^{(e+1)}$ have the same stabilizers in \widetilde{W} . Thus by [8, Thm. 11] (see also [19, Prop. 5.24]) we have an equivalence of categories

$$\mathcal{O}_{\pi^n(1_\mu),R} \simeq \mathcal{O}_{\pi^n(1_{\bar{\mu}}),\bar{R}}, \quad \Delta_R^{w\pi^n(1_\mu)} \mapsto \Delta_{\bar{R}}^{w\pi^n(1_{\bar{\mu}})} \quad \forall w \in \widetilde{W}.$$

Taking the sum by all $n \in \mathbb{Z}$ we get an equivalence of categories

$$\theta_\mu^\mu: O_{\mu,R} \simeq O_{\bar{\mu},\bar{R}}, \quad \Delta_R^{w(1_\mu)} \mapsto \Delta_{\bar{R}}^{w(1_{\bar{\mu}})} \quad \forall w \in \widehat{W}.$$

Recall that we have $\Upsilon(1_\mu) = 1_{\bar{\mu}}$. Thus by \widehat{W} -invariance of Υ we get

$$\theta_\mu^\mu(\Delta_R^\lambda) \simeq \Delta_{\bar{R}}^{\Upsilon(\lambda)} \quad \forall \lambda \in P[\mu].$$

□

Remark 3.13. Notice that [8] yields an equivalence of categories over a field. It is explained in [19] how to get from it an equivalence of categories $O_{\bar{R}}^\Delta$. First, comparing the endomorphisms of projective generators one gets an equivalence of the abelian categories O_R . Then, comparing the highest weight structure in both sides, we deduce an equivalence of additive categories O_R^Δ .

The equivalence θ_μ^μ restricts to equivalences $O_{\mu,R}^\nu \simeq O_{\bar{\mu},\bar{R}}^\nu$ and $O_{\mu,R}^{\nu,\Delta} \simeq O_{\bar{\mu},\bar{R}}^{\nu,\Delta}$ for each parabolic type ν , see [19, Sec. 5.7.2]. We will also call this equivalence θ_μ^μ . We obtain equivalences of categories $\theta_{\mu'}^{\mu'}: O_{\bar{\mu}',\bar{R}}^\nu \simeq O_{\mu',R}^\nu$ and $\theta_{\mu'}^{\mu'}: O_{\bar{\mu}',\bar{R}}^{\nu,\Delta} \simeq O_{\mu',R}^{\nu,\Delta}$ in a similar way.

Conjecture 3.14. *There are the following commutative diagrams*

$$\begin{array}{ccccc} O_{\bar{\mu},\bar{R}}^{\nu,\Delta} & \xrightarrow{\bar{F}_k} & O_{\bar{\mu}^0,\bar{R}}^{\nu,\Delta} & \xrightarrow{\bar{F}_{k+1}} & \bar{O}_{\mu',\bar{R}}^{\nu,\Delta} \\ \uparrow \theta_\mu^\mu & & & & \downarrow \theta_{\mu'}^{\mu'} \\ O_{\mu,R}^{\nu,\Delta} & \xrightarrow{F_k} & & & O_{\mu',R}^{\nu,\Delta} \end{array}$$

and

$$\begin{array}{ccccc}
O_{\overline{\mu}, \overline{R}}^{\nu, \Delta} & \xleftarrow{\overline{E}_k} & O_{\overline{\mu}', \overline{R}}^{\nu, \Delta} & \xleftarrow{\overline{E}_{k+1}} & O_{\overline{\mu}'', \overline{R}}^{\nu, \Delta} \\
\theta_{\overline{\mu}}^{\mu} \downarrow & & & & \uparrow \theta_{\overline{\mu}'}^{\mu'} \\
O_{\mu, R}^{\nu, \Delta} & \xleftarrow{E_k} & O_{\mu', R}^{\nu, \Delta} & &
\end{array}$$

3.7 The commutativity in the Grothendieck groups

We have the following commutative diagram of vector spaces

$$\begin{array}{ccc}
[O_{-e-1, \overline{R}}^{\nu, \Delta}] & \longrightarrow & \wedge^{\nu} U_{e+1} \\
\oplus_{\mu} \theta_{\mu}^{\overline{\mu}} \uparrow & & \uparrow \\
[O_{-e, R}^{\nu, \Delta}] & \longrightarrow & \wedge^{\nu} U_e,
\end{array}$$

where the horizontal maps are respectively the isomorphisms of $\tilde{\mathfrak{sl}}_e$ -modules and $\tilde{\mathfrak{sl}}_{e+1}$ -modules from Proposition 3.10 (e), the right vertical map is given by the injection $U_e \rightarrow U_{e+1}$ in Section 3.3. Moreover, the right vertical map is a morphism of $\tilde{\mathfrak{sl}}_e$ -modules where $\wedge^{\nu} U_{e+1}$ is viewed as an $\tilde{\mathfrak{sl}}_e$ -module via the inclusion $\tilde{\mathfrak{sl}}_e \subset \tilde{\mathfrak{sl}}_{e+1}$ introduced in Section 3.3. Thus $\oplus_{\mu} \theta_{\mu}^{\overline{\mu}}: [O_{-e, R}^{\nu, \Delta}] \rightarrow [O_{-e-1, \overline{R}}^{\nu, \Delta}]$ is a morphism of $\tilde{\mathfrak{sl}}_e$ -modules which intertwines

- $[E_r]$ with $[\overline{E}_r]$, $[F_r]$ with $[\overline{F}_r]$ if $r \in [1, k-1]$,
- $[E_k]$ with $[\overline{E}_k \overline{E}_{k+1}] - [\overline{E}_{k+1} \overline{E}_k]$, $[F_k]$ with $[\overline{F}_{k+1} \overline{F}_k] - [\overline{F}_k \overline{F}_{k+1}]$,
- $[E_r]$ with $[\overline{E}_{r+1}]$, $[F_r]$ with $[\overline{F}_{r+1}]$ if $r \in [k+1, e-1]$.

In particular, we see that the diagrams from Conjecture 3.14 commute at the level of Grothendieck groups. Since there is no good notion of projective functors in the affine category \mathcal{O} , this is not enough to prove our conjecture.

3.8 Partitions

A *partition* of an integer $n \geq 0$ is a tuple of positive integers $(\lambda_1, \dots, \lambda_s)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s$ and $\sum_{t=1}^s \lambda_t = n$. Denote by \mathcal{P}_n the set of all partitions of n and set $\mathcal{P} = \coprod_{n \in \mathbb{N}} \mathcal{P}_n$. For a partition $\lambda = (\lambda_1, \dots, \lambda_s)$ of n , we set $|\lambda| = n$ and $\ell(\lambda) = s$. An *l -partition* of an integer $n \geq 0$ is an l -tuple $\lambda = (\lambda^1, \dots, \lambda^l)$ of partitions of integers $n_1, \dots, n_l \geq 0$ such that $\sum_{t=1}^l n_t = n$. Let \mathcal{P}_n^l be the set of all l -partitions of n and set $\mathcal{P}^l = \coprod_{n \in \mathbb{N}} \mathcal{P}_n^l$.

A partition λ can be represented by a Young diagram $Y(\lambda)$ and an l -partition $\lambda = (\lambda^1, \dots, \lambda^l)$ by an l -tuple of Young diagrams $Y(\lambda) = (Y(\lambda^1), \dots, Y(\lambda^l))$.

Let $\lambda \in \mathcal{P}^l$ be an l -partition. For a box $b \in Y(\lambda)$ situated in the i th row, j th column of the r th component we define its *residue* $\text{Res}_\nu(b) \in I$ as $\nu_r + j - i \pmod e$ and its *deformed residue* $\widetilde{\text{Res}}_\nu(b) \in \widetilde{I}$ as $(\nu_r + j - i, r)$. Set

$$\text{Res}_\nu(\lambda) = \sum_{b \in Y(\lambda)} \alpha_{\text{Res}_\nu(b)} \in Q_I^+, \quad \widetilde{\text{Res}}_\nu(\lambda) = \sum_{b \in Y(\lambda)} \widetilde{\alpha}_{\widetilde{\text{Res}}_\nu(b)} \in Q_{\widetilde{I}}^+.$$

Now for $\alpha \in Q_I^+$ and $\widetilde{\alpha} \in Q_{\widetilde{I}}^+$ set

$$\mathcal{P}_\alpha^l = \{\lambda \in \mathcal{P}^l; \text{Res}_\nu(\lambda) = \alpha\}, \quad \mathcal{P}_{\widetilde{\alpha}}^l = \{\lambda \in \mathcal{P}^l; \widetilde{\text{Res}}_\nu(\lambda) = \widetilde{\alpha}\}.$$

This notation depends on ν . We may write $\mathcal{P}_{\alpha,\nu}^l$ and $\mathcal{P}_{\widetilde{\alpha},\nu}^l$ to specify ν . We have decompositions

$$\mathcal{P}_d^l = \bigoplus_{\alpha \in Q_I^+, |\alpha|=d} \mathcal{P}_\alpha^l, \quad \mathcal{P}_\alpha^l = \bigoplus_{\widetilde{\alpha} \in Q_{\widetilde{I}}^+, \pi_e(\widetilde{\alpha})=\alpha} \mathcal{P}_{\widetilde{\alpha}}^l.$$

3.9 The category \mathbf{A}

Let $\mathcal{P}_d^\nu \subset \mathcal{P}_d^l$ be the subset of the elements $\lambda = (\lambda^1, \dots, \lambda^l)$ such that $\ell(\lambda^r) \leq \nu_r$ for each $r \in [1, l]$. We can view λ as the weight in P given by

$$(\lambda_1^1, \dots, \lambda_{\ell(\lambda^1)}^1, 0^{m_1 - \ell(\lambda^1)}, \lambda_1^2, \dots, \lambda_{\ell(\lambda^2)}^2, 0^{m_2 - \ell(\lambda^2)}, \dots, \lambda_1^l, \dots, \lambda_{\ell(\lambda^l)}^l, 0^{m_l - \ell(\lambda^l)}).$$

Then, we set $\omega(\lambda) = \lambda - \rho + \rho_\nu$ in P .

Definition 3.15. Let $\mathbf{A}_R^\nu[d] \subset O_{-e,R}^\nu$ be the Serre subcategory generated by the modules $\Delta(\omega(\lambda))_R$ with $\lambda \in \mathcal{P}_d^\nu$, see Section 3.1. Denote by $\mathbf{A}_R^{\nu,\Delta}[d]$ the full subcategory of Δ -filtered modules in $\mathbf{A}_R^\nu[d]$.

We abbreviate $\Delta[\lambda]_R = \Delta(\omega(\lambda))_R$. The restriction of the functor F to the subcategory $\mathbf{A}_R^{\nu,\Delta}[d]$ yields a functor $F: \mathbf{A}_R^{\nu,\Delta}[d] \rightarrow \mathbf{A}_R^{\nu,\Delta}[d+1]$. However, it is not true that $E(\mathbf{A}_R^{\nu,\Delta}[d+1]) \subset \mathbf{A}_R^{\nu,\Delta}[d]$. Nevertheless, we can define a functor $E: \mathbf{A}_R^{\nu,\Delta}[d+1] \rightarrow \mathbf{A}_R^{\nu,\Delta}[d]$ that is left adjoint to $F: \mathbf{A}_R^{\nu,\Delta}[d] \rightarrow \mathbf{A}_R^{\nu,\Delta}[d+1]$, see [19, Sec. 5.9]. This can be done in the following way. Let h be the inclusion functor from $\mathbf{A}_R^{\nu,\Delta}[d]$ to $O_{-e,R}^{\nu,\Delta}$. Abusing the notation, we will use the same symbol for the inclusion functor from $\mathbf{A}^{\nu,\Delta}[d+1]$ to $O_{-e,R}^{\nu,\Delta}$. Let h^* be the left adjoint functor to h . We define the functor E for the category $\mathbf{A}_R^{\nu,\Delta}$ as $h^* E h$.

There is a decomposition $\mathbf{A}_R^\nu[d] = \bigoplus_{\alpha \in Q_I^+, |\alpha|=d} \mathbf{A}_R^\nu[\alpha]$, where $\mathbf{A}_R^\nu[\alpha]$ is the Serre subcategory of $\mathbf{A}_R^\nu[d]$ generated by the Verma modules $\Delta[\lambda]_R$ such that $\lambda \in \mathcal{P}_\alpha^l$. The functors E, F admit decompositions

$$E = \bigoplus_{i \in I} E_i, \quad F = \bigoplus_{i \in I} F_i$$

such that for each $\alpha \in Q_I^+$ and $i \in I$ we have

$$E_i(\mathbf{A}_R^{\nu,\Delta}[\alpha]) \subset \mathbf{A}_R^{\nu,\Delta}[\alpha - \alpha_i], \quad F_i(\mathbf{A}_R^{\nu,\Delta}[\alpha]) \subset \mathbf{A}_R^{\nu,\Delta}[\alpha + \alpha_i].$$

Note that the functor E_i for the category $\mathbf{A}_R^{\nu, \Delta}$ is the restriction of the functor E_i for the category $O_{-e, R}^{\nu, \Delta}$ if $i \neq 0$. Thus for $i \neq 0$ the pair of functors (E_i, F_i) for the category $\mathbf{A}_R^{\nu, \Delta}$ is biadjoint. But we have only a one-side adjunction (E_0, F_0) . Note also that if R is a field, then we can define the functors as above (with the same adjunction properties) for the category \mathbf{A}_R^ν instead of $\mathbf{A}_R^{\nu, \Delta}$.

3.10 The change of level for \mathbf{A}

For $\lambda_1, \lambda_2 \in P$ we write $\lambda_1 \geq \lambda_2$ if $(\lambda_1)_r \geq (\lambda_2)_r$ for each $r \in [1, N]$. Here, $(\lambda_i)_r$ is the r th entry of λ_i for each r . We identify Q_I with a sublattice of X_I^δ via the map ι^δ defined in Section 2.1.

Lemma 3.16. (a) For each $\lambda_1, \lambda_2 \in P$ we have $\mathbf{wt}_e^\delta(\lambda_1) - \mathbf{wt}_e^\delta(\lambda_2) \in Q_I$.
(b) If we also have $\lambda_1 \leq \lambda_2$, then $\mathbf{wt}_e^\delta(\lambda_1) - \mathbf{wt}_e^\delta(\lambda_2) \in Q_I^+$.

Proof. It is enough to assume that we have $\lambda_1 = \lambda_2 - \epsilon_r$ for some $r \in [1, N]$. In this case we have $\mathbf{wt}_e^\delta(\lambda_1) - \mathbf{wt}_e^\delta(\lambda_2) = \alpha_i$, where $i \in I$ is the residue of the integer $(\lambda_1)_r$ modulo e . \square

Let us write \emptyset for the empty l -partition. Note that $\Delta[\emptyset]_R = \Delta_R^{\rho_\nu}$ is the Verma module of highest weight $\widetilde{\rho_\nu - \rho}$. Since, ρ_ν lies in $P[\mathbf{wt}_e(\rho_\nu)]$, we have $\Delta[\emptyset]_R \in O_{\mathbf{wt}_e(\rho_\nu), R}^\nu$. More generally, fix an element $\alpha = \sum_{i \in I} d_i \alpha_i$ in Q_I^+ . Put $\mu = \mathbf{wt}_e(\rho_\nu) - \alpha \in X_I$. See [19, Sec. 2.3] for the definition of a highest weight category over a local ring. The following proposition holds, see [19, Sec. 5.5].

Proposition 3.17. The category $\mathbf{A}_R^\nu[\alpha]$ is a full subcategory of $O_{\mu, R}^\nu$ that is a highest weight category. \square

For $\lambda \in \mathcal{P}_d^l$ let $P[\lambda]_R$, $\nabla[\lambda]_R$ and $T[\lambda]_R$ be the projective, costandard and the tilting objects in $\mathbf{A}_R^\nu[d]$ with parameter λ , see [19, Prop. 2.1].

Set $\bar{\alpha} = \phi(\alpha) \in Q_I$, $\bar{\mu} = \phi(\mu) \in X_I$ and $\beta = \mathbf{wt}_{e+1}^\delta(\rho_\nu) - \mathbf{wt}_{e+1}^\delta(\Upsilon(\rho_\nu))$. By Lemma 3.16, we have $\beta \in Q_I^+$.

Proposition 3.18. The equivalence of categories $\theta_\mu^{\bar{\mu}}$ takes the subcategory $\mathbf{A}_R^\nu[\alpha]$ of $O_{\mu, R}^\nu$ to the subcategory $\mathbf{A}_{\bar{\mu}, \bar{R}}^\nu[\beta + \bar{\alpha}]$ of $O_{\bar{\mu}, \bar{R}}^\nu$. \square

Let the map $\phi: Q_I \rightarrow Q_I$ be as in Section 2.2 (see also Section 2.8) and Υ be as in (10). First, we prove the following lemma.

Lemma 3.19. If $\lambda_1, \lambda_2 \in P$, then

$$\mathbf{wt}_{e+1}^\delta(\Upsilon(\lambda_1)) - \mathbf{wt}_{e+1}^\delta(\Upsilon(\lambda_2)) = \phi(\mathbf{wt}_e^\delta(\lambda_1) - \mathbf{wt}_e^\delta(\lambda_2)).$$

Proof of Lemma 3.19. It is enough to prove the statement in the case where we have $\lambda_1 = \lambda_2 - \epsilon_r$ for some $r \in [1, N]$. In this case we have $\mathbf{wt}_e^\delta(\lambda_1) - \mathbf{wt}_e^\delta(\lambda_2) = \alpha_i$, where i is the residue of $(\lambda_1)_r$ modulo e . If $i \neq k$ then we have $\mathbf{wt}_{e+1}^\delta(\Upsilon(\lambda_1)) - \mathbf{wt}_{e+1}^\delta(\Upsilon(\lambda_2)) = \bar{\alpha}_{i'} = \phi(\alpha_i)$, where i' is as in (5). If $i = k$ then we have $\mathbf{wt}_{e+1}^\delta(\Upsilon(\lambda_1)) - \mathbf{wt}_{e+1}^\delta(\Upsilon(\lambda_2)) = \bar{\alpha}_k + \bar{\alpha}_{k+1} = \phi(\alpha_k)$. \square

Proof of Proposition 3.18. By definition, $\mathbf{A}_R^\nu[\alpha] \subset O_{\mu,R}^\nu$ is the Serre subcategory of $O_{\mu,R}^\nu$ generated by Δ_R^λ such that the weight $\lambda \in P^\nu$ satisfies $\lambda \geq \rho_\nu$ and $\mathbf{wt}_e^\delta(\rho_\nu) - \mathbf{wt}_e^\delta(\lambda) = \alpha$. Here \geq is the order defined before Lemma 3.16.

As $\theta_\mu^\mu(\Delta_R^\lambda)$ is isomorphic to $\Delta_R^{\Upsilon(\lambda)}$, Lemma 3.19 implies that $\theta_\mu^\mu(\mathbf{A}_R^\nu[\alpha])$ is the Serre subcategory of $O_{\mu,R}^\nu$ generated by $\Delta_R^{\bar{\lambda}}$ for $\bar{\lambda} \in P^\nu$ such that $\bar{\lambda} \geq \Upsilon(\rho_\nu)$ and $\mathbf{wt}_{e+1}^\delta(\Upsilon(\rho_\nu)) - \mathbf{wt}_{e+1}^\delta(\bar{\lambda}) = \bar{\alpha}$.

Moreover, for each module $\Delta_R^{\bar{\lambda}} \in O_{\mu,\bar{R}}^\nu$, the weight $\bar{\lambda}$ has no coordinates that are congruent to $k+1$ modulo $e+1$. Then $\bar{\lambda}$ satisfies $\bar{\lambda} \geq \rho_\nu$ if and only if it satisfies $\bar{\lambda} \geq \Upsilon(\rho_\nu)$. We have $\mathbf{wt}_{e+1}^\delta(\rho_\nu) - \mathbf{wt}_{e+1}^\delta(\Upsilon(\rho_\nu)) = \beta$. Thus $\theta_\mu^\mu(\mathbf{A}_R^\nu[\alpha])$ is the Serre subcategory of $O_{\mu,\bar{R}}^\nu$ generated by the modules $\Delta_R^{\bar{\lambda}}$ where $\bar{\lambda}$ runs over the set of all $\bar{\lambda} \in P^\nu$ such that $\bar{\lambda} \geq \rho_\nu$ and $\mathbf{wt}_{e+1}^\delta(\rho_\nu) - \mathbf{wt}_{e+1}^\delta(\bar{\lambda}) = \bar{\alpha} + \beta$. This implies $\theta_\mu^\mu(\mathbf{A}_R^\nu[\alpha]) = \mathbf{A}_{\bar{R}}^\nu[\bar{\alpha} + \beta]$. \square

3.11 The category \mathcal{A}

From now on, to avoid cumbersome notation we will use the following abbreviations. First, for each $\alpha \in Q_I^+$ we set

$$\mathcal{A}_R^\nu[\alpha] = \mathbf{A}_R^\nu[\beta + \bar{\alpha}], \quad \mathcal{A}_R^\nu[d] = \bigoplus_{|\alpha|=d} \mathcal{A}_R^\nu[\alpha], \quad \mathcal{A}_R^\nu = \bigoplus_{d \in \mathbb{N}} \mathcal{A}_R^\nu[d].$$

Next, we define the endofunctors $E_0, \dots, E_{e-1}, F_0, \dots, F_{e-1}$ of $\mathcal{A}_R^{\nu,\Delta}$ (or of \mathcal{A}_R^ν if R is a field) by

$$\begin{aligned} F_0 &= \bar{F}_0|_{\mathcal{A}_R^{\nu,\Delta}}, \dots, F_{k-1} = \bar{F}_{k-1}|_{\mathcal{A}_R^{\nu,\Delta}}, \quad F_k = \bar{F}_{k+1}\bar{F}_k|_{\mathcal{A}_R^{\nu,\Delta}}, \\ F_{k+1} &= \bar{F}_{k+2}|_{\mathcal{A}_R^{\nu,\Delta}}, \dots, F_{e-1} = \bar{F}_e|_{\mathcal{A}_R^{\nu,\Delta}}, \\ E_0 &= \bar{E}_0|_{\mathcal{A}_R^{\nu,\Delta}}, \dots, E_{k-1} = \bar{E}_{k-1}|_{\mathcal{A}_R^{\nu,\Delta}}, \quad E_k = \bar{E}_k\bar{F}_{k+1}|_{\mathcal{A}_R^{\nu,\Delta}}, \\ E_{k+1} &= \bar{E}_{k+2}|_{\mathcal{A}_R^{\nu,\Delta}}, \dots, E_{e-1} = \bar{E}_e|_{\mathcal{A}_R^{\nu,\Delta}}. \end{aligned} \tag{26}$$

By definition, we have $E_i(\mathcal{A}_R^{\nu,\Delta}[\alpha]) \subset \mathcal{A}_R^{\nu,\Delta}[\alpha - \alpha_i]$ and $F_i(\mathcal{A}_R^{\nu,\Delta}[\alpha]) \subset \mathcal{A}_R^{\nu,\Delta}[\alpha + \alpha_i]$. Consider the endofunctors $E = \bigoplus_{i \in I} E_i$ and $F = \bigoplus_{i \in I} F_i$ of $\mathcal{A}_R^{\nu,\Delta}$. We have $E(\mathcal{A}_R^{\nu,\Delta}[d]) \subset \mathcal{A}_R^{\nu,\Delta}[d-1]$ and $F(\mathcal{A}_R^{\nu,\Delta}[d]) \subset \mathcal{A}_R^{\nu,\Delta}[d+1]$.

Let $\theta_\alpha: \mathbf{A}_R^\nu[\alpha] \rightarrow \mathcal{A}_R^\nu[\alpha]$ be the equivalence of categories in Proposition 3.18. Taking the sum over α 's, we get an equivalence $\theta: \mathbf{A}_R^\nu \rightarrow \mathcal{A}_R^\nu$ and an equivalence $\theta_d: \mathbf{A}_R^\nu[d] \rightarrow \mathcal{A}_R^\nu[d]$. Moreover, we have the following commutative diagram of Grothendieck groups

$$\begin{array}{ccc} [\mathbf{A}_R^{\nu,\Delta}] & \xrightarrow{F_i} & [\mathbf{A}_R^{\nu,\Delta}] \\ \theta \downarrow & & \theta \downarrow \\ [\mathcal{A}_R^{\nu,\Delta}] & \xrightarrow{F_i} & [\mathcal{A}_R^{\nu,\Delta}], \end{array} \tag{27}$$

see Section 3.7.

For $\lambda \in \mathcal{P}_d^l$ we set $\overline{\Delta}[\lambda]_R = \Delta_R^{\Upsilon(\rho_\nu + \lambda)} \in \mathcal{A}_R^\nu[d]$. By construction we have $\theta_d(\Delta[\lambda]_R) \simeq \overline{\Delta}[\lambda]_R$. Let $\overline{P}_R[\lambda]$, $\overline{V}_R[\lambda]$ and $\overline{T}_R[\lambda]$ be the projective, the costandard and the tilting object with parameter λ in $\mathcal{A}_R^\nu[d]$.

3.12 The categorical representation in the category \mathcal{O} over the field K

In this section we compare the categorical representation in $O_{-e,K}^\nu$ with the representation datum in $O_{-e,R}^{\nu,\Delta}$ introduced above. First, for each $\lambda \in P$ we define the following weight in X_I^+

$$\widetilde{\mathbf{wt}}_e(\lambda) = \sum_{r=1}^l \left(\sum_{t=\nu_1+\dots+\nu_{r-1}+1}^{\nu_1+\dots+\nu_r} \widetilde{\varepsilon}_{(\lambda_t, r)} \right).$$

For $\tilde{\mu} \in X_I^+$ let $O_{\tilde{\mu},K}^\nu$ be the Serre subcategory of $O_{-e,K}^\nu$ generated by the Verma modules Δ_K^λ such that $\widetilde{\mathbf{wt}}_e(\lambda) = \tilde{\mu}$. This decomposition is a refinement of the decomposition $O_{-e,K}^\nu = \bigoplus_{\mu \in X_I} O_{\mu,K}^\nu$ introduced in Section 3.6. More precisely, we have

$$O_{\mu,K}^\nu = \bigoplus_{\tilde{\mu} \in X_I^+, \pi_e(\tilde{\mu}) = \mu} O_{\tilde{\mu},K}^\nu.$$

Similarly, there are decompositions

$$E = \bigoplus_{j \in \tilde{I}} E_j, \quad F = \bigoplus_{j \in \tilde{I}} F_j \quad (28)$$

such that E_j and F_j map $O_{\tilde{\mu},K}$ to $O_{\tilde{\mu}+\tilde{\alpha}_j,K}$ and $O_{\tilde{\mu}-\tilde{\alpha}_j,K}$ respectively. We set

$$E_i = \bigoplus_{j \in \tilde{I}, \pi_e(j)=i} E_j, \quad F_i = \bigoplus_{j \in \tilde{I}, \pi_e(j)=i} F_j.$$

We have commutative diagrams

$$\begin{array}{ccc} O_{-e,R}^{\nu,\Delta} & \xrightarrow{E_i} & O_{-e,R}^{\nu,\Delta} \\ K \otimes_R \bullet \downarrow & & K \otimes_R \bullet \downarrow \\ O_{-e,K}^{\nu,\Delta} & \xrightarrow{E_i} & O_{-e,K}^{\nu,\Delta} \end{array} \quad \begin{array}{ccc} O_{-e,R}^{\nu,\Delta} & \xrightarrow{F_i} & O_{-e,R}^{\nu,\Delta} \\ K \otimes_R \bullet \downarrow & & K \otimes_R \bullet \downarrow \\ O_{-e,K}^{\nu,\Delta} & \xrightarrow{F_i} & O_{-e,K}^{\nu,\Delta} \end{array}$$

For each element $\tilde{\alpha} \in Q_I^+$ let $\mathbf{A}_K^\nu[\tilde{\alpha}]$ be the Serre subcategory of \mathbf{A}_K^ν generated by the Verma modules $\Delta[\lambda]_K$ such that $\lambda \in \mathcal{P}_{\tilde{\alpha},\nu}^l$. Similarly to Section 3.9, we have

$$E_j(\mathbf{A}_K^\nu[\tilde{\alpha}]) \subset \mathbf{A}_K^\nu[\tilde{\alpha} - \tilde{\alpha}_j], \quad F_j(\mathbf{A}_K^\nu[\tilde{\alpha}]) \subset \mathbf{A}_K^\nu[\tilde{\alpha} + \tilde{\alpha}_j].$$

See [19, Sec. 7.4] for details.

Similarly, for $j \in \tilde{I}$ we can define the endofunctor $\overline{E}_j, \overline{F}_j$ of the categories $O_{-e-1, \overline{\mathbf{k}}}^\nu$ and $\overline{\mathbf{A}}_{\overline{\mathbf{k}}}^\nu$.

The decomposition $\mathbf{A}_K^\nu[\alpha] = \bigoplus_{\pi_e(\tilde{\alpha})=\alpha} \mathbf{A}_K^\nu[\tilde{\alpha}]$ yields a decomposition $\mathcal{A}_K^\nu[\alpha] = \bigoplus_{\pi_e(\tilde{\alpha})=\alpha} \mathcal{A}_K^\nu[\tilde{\alpha}]$. We also consider the endofunctors E_j, F_j of \mathcal{A}_K^ν such that for $j = (a, b) \in \tilde{I}$ we have the following analogue of (26):

$$E_j = \begin{cases} \overline{E}_{(\Upsilon(a), b)}|_{\mathcal{A}_K^\nu} & \text{if } \pi_e(j) \neq k, \\ \overline{E}_{(\Upsilon(a), b)}\overline{E}_{(\Upsilon(a)+1, b)}|_{\mathcal{A}_K^\nu} & \text{if } \pi_e(j) = k, \end{cases}$$

$$F_j = \begin{cases} \overline{F}_{(\Upsilon(a), b)}|_{\mathcal{A}_K^\nu} & \text{if } \pi_e(j) \neq k, \\ \overline{F}_{(\Upsilon(a)+1, b)}\overline{F}_{(\Upsilon(a), b)}|_{\mathcal{A}_K^\nu} & \text{if } \pi_e(j) = k. \end{cases}$$

We have $E_j(\mathcal{A}_K^\nu[\tilde{\alpha}]) \subset \mathcal{A}_K^\nu[\tilde{\alpha} - \tilde{\alpha}_j]$ and $F_j(\mathcal{A}_K^\nu[\tilde{\alpha}]) \subset \mathcal{A}_K^\nu[\tilde{\alpha} + \tilde{\alpha}_j]$.

3.13 The modules $T_{\alpha, R}, \overline{T}_{\alpha, R}$

Consider the module $T_{\alpha, R} = F_\alpha(\Delta[\emptyset]_R)$ in $\mathbf{A}_R^\nu[\alpha]$ and the module $\overline{T}_{\alpha, R} = F_\alpha(\overline{\Delta}[\emptyset]_R)$ in $\mathcal{A}_R^\nu[\alpha]$. The commutativity of the diagram (27) implies that we have the following equality of classes $[\theta_\alpha(T_{\alpha, R})] = [\overline{T}_{\alpha, R}]$ in $[\mathcal{A}_R^\nu]$. The modules $T_{\alpha, R} \in \mathbf{A}_R^\nu[\alpha]$ and $\overline{T}_{\alpha, R} \in \mathcal{A}_R^\nu[\alpha]$ are tilting because $\Delta[\emptyset]_R \in \mathbf{A}_R^\nu[0]$ and $\overline{\Delta}[\emptyset]_R \in \mathcal{A}_R^\nu[0]$ are tilting and the functors F and \overline{F} preserve tilting modules, see [19, Lem. 8.33, Lem. 5.16 (b)]. Since a tilting module is characterized by its class in the Grothendieck group, we deduce that there is an isomorphism of modules

$$\theta_\alpha(T_{\alpha, R}) \simeq \overline{T}_{\alpha, R}. \quad (29)$$

From now on, we assume that $\nu_r \geq |\alpha|$ for each $r \in [1, l]$. Consider the weight

$$\Lambda_\nu := \sum_{r=1}^l \Lambda_{\nu_r} \in P_I. \quad (30)$$

We may abbreviate $\Lambda = \Lambda_\nu$. Assume that $R = \mathbf{k}$. The following result is proved in [19, Theorem 5.37].

Proposition 3.20. *The homomorphism $R_{\alpha, \mathbf{k}} \rightarrow \text{End}(F_\alpha(\Delta[\emptyset]_{\mathbf{k}}))^{\text{op}}$ induced by the categorical representation of $\tilde{\mathfrak{sl}}_e$ in $O_{-e, \mathbf{k}}^\nu$ yields an isomorphism $\psi_{\alpha, \mathbf{k}}: R_{\alpha, \mathbf{k}}^\Lambda \simeq \text{End}(T_{\alpha, \mathbf{k}})^{\text{op}}$. \square*

Consider the category $\overline{\mathcal{C}} = O_{-e-1, \overline{\mathbf{k}}}^\nu$. Now we must construct a similar isomorphism $\overline{\psi}_{\alpha, \mathbf{k}}: R_{\alpha, \mathbf{k}}^\Lambda \simeq \text{End}(\overline{T}_{\alpha, \mathbf{k}})^{\text{op}}$ coming from the $\tilde{\mathfrak{sl}}_{e+1}$ -categorical representation in $\overline{\mathcal{C}}$.

Lemma 3.9 yields a categorical representation of $\tilde{\mathfrak{sl}}_e$ in a subcategory \mathcal{C} of $\overline{\mathcal{C}}$. As we have $\overline{\Delta}[\emptyset]_R \in \mathcal{C}$, there is an algebra homomorphism

$$R_{\alpha, \mathbf{k}} \rightarrow \text{End}(\overline{T}_{\alpha, \mathbf{k}})^{\text{op}}. \quad (31)$$

Lemma 3.21. *The homomorphism (31) factors through a homomorphism $\overline{\psi}_{\alpha, \mathbf{k}}: R_{\alpha, \mathbf{k}}^{\Lambda} \rightarrow \text{End}(\overline{T}_{\alpha, \mathbf{k}})^{\text{op}}$.*

Proof. The statement follows from Lemma 3.22 below applied to $M = \overline{\Delta}[\emptyset]_{\mathbf{k}}$. \square

Lemma 3.22. *Let $\mathcal{C} = \bigoplus_{\mu \in X_I} \mathcal{C}_{\mu}$ be a categorical representation of \mathfrak{g}_I over \mathbf{k} . Let $M \in \mathcal{C}$ such that there are non-negative integers t_i for $i \in I$ such that*

- $\text{End}(M) \simeq \mathbf{k}$,
- $E_i F_i(M) \simeq M^{\oplus t_i}, \quad \forall i \in I$.

Then, for each $d \in \mathbb{N}$, the homomorphism $R_{d, \mathbf{k}} \rightarrow \text{End}(F^d(M))^{\text{op}}$ factors through the cyclotomic quotient with respect to the weight $\sum_{i \in I} t_i \Lambda_i$.

Proof. It suffices to prove the statement for $d = 1$. By adjointness we have the following isomorphisms of vector spaces

$$\text{Hom}(F_i(M), F_i(M)) \simeq \text{Hom}(E_i F_i(M), M) \simeq \text{Hom}(M, M)^{\oplus t_i} \simeq \mathbf{k}^{t_i}.$$

The image of x in $\text{End}(F_i(M))$ is nilpotent. Thus it must be killed by the t_i th power because $\dim \text{End}(F_i(M)) = t_i$. \square

Remark 3.23. The lemma above admits the following equivalent version.

Let $\mathcal{C} = \bigoplus_{\mu \in X_{\mathcal{F}}} \mathcal{C}_{\mu}$ be a categorical representation of $\mathfrak{g}_{\mathcal{F}}$ over \mathbf{k} . Let $M \in \mathcal{C}$ be an object such that there are non-negative integers t_i for $i \in \mathcal{F}$ such that t_i is non-zero only for finitely many i and

- $\text{End}(M) \simeq \mathbf{k}$,
- $E_i F_i(M) \simeq M^{\oplus t_i}, \quad \forall i \in \mathcal{F}$.

For each $d \in \mathbb{N}$, the homomorphism $H_{d, \mathbf{k}}(q) \rightarrow \text{End}(F^d(M))^{\text{op}}$ factors through the cyclotomic quotient $H_{d, \mathbf{k}}^Q(q)$, where Q is a tuple of elements of \mathcal{F} such that Q contains t_i copies of the element i for each $i \in \mathcal{F}$.

Now we are going to prove that the algebra homomorphism $\overline{\psi}_{\alpha, \mathbf{k}}$ constructed in Lemma 3.21 is an isomorphism. To do this, we will use a deformation argument. Recall from Corollary 2.18 that the cyclotomic KLR algebra $R_{\alpha, \mathbf{k}}^{\Lambda}$ is isomorphic to the cyclotomic Hecke algebra $H_{\alpha, \mathbf{k}}^{\nu}(\zeta_e)$. We will use the algebra $H_{\alpha, R}^{\nu}(q_e)$ as an R -version of $R_{\alpha, \mathbf{k}}^{\Lambda}$.

Remark 3.24. The homomorphism $H_{d, \overline{R}}(q_{e+1}) \rightarrow \text{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_R))^{\text{op}}$ coming from the representation datum structure commutes with base changes $\mathbf{k} \otimes_R \bullet$ and $K \otimes_R \bullet$, see [19, Prop. 8.30].

Lemma 3.25. *For each $d \in \mathbb{N}$ the homomorphism $H_{d,\overline{R}}(q_{e+1}) \rightarrow \text{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_R))^{\text{op}}$ extends to the algebra $\widehat{H}_{d,\overline{R}}(q_{e+1})$ in such a way that the idempotent $e(\mathbf{i})$ goes to the projector to $\overline{F}_{\mathbf{i}}(\overline{\Delta}[\emptyset]_R)$ for each \mathbf{i} .*

Proof. We must prove that for each $\mathbf{i} \in \overline{T}^d$ we have the following.

- (a₁) The element $(X_r - X_t)e(\mathbf{i}) \in H_{d,\overline{R}}(q_{e+1})$ acts on $F_{\mathbf{i}}(\overline{\Delta}[\emptyset]_R)$ by an invertible operator for each $r \neq t$ such that $i_r \neq i_t$.
- (a₂) The element $(q_{e+1}X_r - X_t)e(\mathbf{i}) \in H_{d,\overline{R}}(q_{e+1})$ acts on $F_{\mathbf{i}}(\overline{\Delta}[\emptyset]_R)$ by an invertible operator for each $r \neq t$ such that $i_r + 1 \neq i_t$.

To prove this we need the following standard lemma.

Lemma 3.26. *Let M be an R -module. Let $\phi: M \rightarrow M$ be an endomorphism of M .*

- (a) *If M is finitely generated over R and the endomorphism $\mathbf{k}\phi$ of $\mathbf{k} \otimes_R M$ is surjective then ϕ is surjective.*
- (b) *If M is free over R and the endomorphism $K\phi$ of $K \otimes_R M$ is injective then ϕ is injective.* \square

We have a commutative diagram

$$\begin{array}{ccc}
H_{d,\overline{\mathbf{k}}}(\zeta_{e+1}) & \longrightarrow & \text{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_{\mathbf{k}}))^{\text{op}} \\
\uparrow & & \uparrow \\
H_{d,\overline{R}}(q_{e+1}) & \longrightarrow & \text{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_R))^{\text{op}} \\
\downarrow & & \downarrow \\
H_{d,\overline{K}}(q_{e+1}) & \longrightarrow & \text{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_K))^{\text{op}}
\end{array}$$

By Remark 3.6 the top and the bottom vertical homomorphisms extend to homomorphisms

$$\widehat{H}_{d,\overline{\mathbf{k}}}(\zeta_{e+1}) \rightarrow \text{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_{\mathbf{k}}))^{\text{op}}$$

and

$$\widehat{H}_{d,\overline{K}}(q_{e+1}) \rightarrow \text{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_K))^{\text{op}}.$$

In particular the elements from (a₁) and (a₂) go to elements of $H_{d,\overline{\mathbf{k}}}(\zeta_{e+1})$ and $H_{d,\overline{K}}(q_{e+1})$ that act on $\overline{F}_{\mathbf{i}}(\overline{\Delta}[\emptyset]_{\mathbf{k}})$ and $\overline{F}_{\mathbf{i}}(\overline{\Delta}[\emptyset]_K)$ by invertible operators. To conclude we apply Lemma 3.26 to each weight space of the R -module $\overline{F}_{\mathbf{i}}(\overline{\Delta}[\emptyset]_R)$ (that is a free R -module of finite rank because $\overline{F}_{\mathbf{i}}(\overline{\Delta}[\emptyset]_R)$ is tilting by [19, Lem. 8.33, Lem. 5.16 (b)]). \square

The homomorphism $\widehat{H}_{d,\overline{R}}(q_{e+1}) \rightarrow \text{End}(\overline{F}^d(\overline{\Delta}[\emptyset]_R))^{\text{op}}$ from Lemma 3.25 yields a homomorphism $\widehat{H}_{\alpha,\overline{R}}(q_{e+1}) \rightarrow \text{End}(\overline{F}_{\alpha}^d(\overline{\Delta}[\emptyset]_R))^{\text{op}}$ for each $\alpha \in Q_I^+$.

Consider the sum $\mathbf{e} = \sum_{\mathbf{i} \in I_{\text{ord}}^{\bar{\alpha}}} e(\mathbf{i})$ in $\widehat{H}_{\bar{\alpha}, \bar{R}}(q_{e+1})$. We get an algebra homomorphism

$$\mathbf{e} \widehat{H}_{\bar{\alpha}, \bar{R}}(q_{e+1}) \mathbf{e} \rightarrow \text{End}(\bar{T}_{\alpha, R})^{\text{op}} \quad (32)$$

because $\bar{T}_{\alpha, R} = \bigoplus_{\mathbf{i} \in I_{\text{ord}}^{\bar{\alpha}}} \bar{F}_{\mathbf{i}}(\bar{\Delta}[\emptyset]_R)$.

Lemma 3.27. *The homomorphism (32) factors through $\widehat{SH}_{\bar{\alpha}, \bar{R}}(q_{e+1})$.*

Proof. We can construct a K -linear version $\mathbf{e} \widehat{H}_{\bar{\alpha}, \bar{K}}(q_{e+1}) \mathbf{e} \rightarrow \text{End}(\bar{T}_{\alpha, K})^{\text{op}}$ of the homomorphism (32). It factors through a homomorphism $\widehat{SH}_{\bar{\alpha}, \bar{K}}(q_{e+1}) \rightarrow \text{End}(\bar{T}_{\alpha, K})^{\text{op}}$ because $\bar{F}_{\mathbf{j}}(\bar{\Delta}[\emptyset]_K) = 0$ for each $\mathbf{j} \in \tilde{I}_{\text{un}}^{\bar{\alpha}}$. Here we set

$$\tilde{I}_{\text{un}}^{\bar{\alpha}} = \coprod_{\tilde{\alpha} \in Q_{\tilde{I}}^+, \pi_{e+1}(\tilde{\alpha}) = \bar{\alpha}} \tilde{I}_{\text{un}}^{\tilde{\alpha}}.$$

Moreover, we have a commutative diagram

$$\begin{array}{ccc} \mathbf{e} \widehat{H}_{\bar{\alpha}, \bar{R}}(q_{e+1}) \mathbf{e} & \longrightarrow & \text{End}(\bar{T}_{\alpha, R})^{\text{op}} \\ \downarrow & & \downarrow \\ \mathbf{e} \widehat{H}_{\bar{\alpha}, \bar{K}}(q_{e+1}) \mathbf{e} & \longrightarrow & \text{End}(\bar{T}_{\alpha, K})^{\text{op}}. \end{array} \quad (33)$$

By definition, the kernel of the homomorphism

$$\mathbf{e} \widehat{H}_{\bar{\alpha}, \bar{R}}(q_{e+1}) \mathbf{e} \rightarrow \widehat{SH}_{\bar{\alpha}, \bar{R}}(q_{e+1})$$

is the intersection of the kernel of the homomorphism

$$\mathbf{e} \widehat{H}_{\bar{\alpha}, \bar{K}}(q_{e+1}) \mathbf{e} \rightarrow \widehat{SH}_{\bar{\alpha}, \bar{K}}(q_{e+1})$$

with $\mathbf{e} \widehat{H}_{\bar{\alpha}, \bar{R}}(q_{e+1}) \mathbf{e}$. This proves the statement because the right vertical map in (33) is injective because the module $\bar{T}_{\alpha, R}$ is tilting. \square

Composing the homomorphism $\widehat{SH}_{\bar{\alpha}, \bar{R}}(q_{e+1}) \rightarrow \text{End}(\bar{T}_{\alpha, R})^{\text{op}}$ in Lemma 3.27 with the homomorphism $\Phi_{\alpha, R}$ in Lemma 2.21 yield an algebra homomorphism

$$\widehat{H}_{\alpha, R}(q_e) \rightarrow \text{End}(\bar{T}_{\alpha, R})^{\text{op}} \quad (34)$$

Note that there is a surjective algebra homomorphism $\widehat{H}_{\alpha, R}(q_e) \rightarrow H_{\alpha, R}^{\nu}(q_e)$ (see also Corollary 2.18).

Lemma 3.28. *The homomorphism (34) factors through a homomorphism $\bar{\psi}_{\alpha, R}: H_{\alpha, R}^{\nu}(q_e) \rightarrow \text{End}(\bar{T}_{\alpha, R})^{\text{op}}$.*

Proof. The proof is similar to the proof of Lemma 3.27. The K -linear version $\widehat{H}_{\bar{\alpha},K}(q_e) \rightarrow \text{End}(\overline{T}_{\alpha,K})^{\text{op}}$ of this homomorphism factors through the cyclotomic quotient $H_{d,K}^{\nu}(q_e)$ by Remark 3.23. Then we deduce the statement from the commutativity of the following diagram

$$\begin{array}{ccc} \widehat{H}_{\bar{\alpha},R}(q_e) & \longrightarrow & \text{End}(\overline{T}_{\alpha,R})^{\text{op}} \\ \downarrow & & \downarrow \\ \widehat{H}_{\bar{\alpha},K}(q_e) & \longrightarrow & \text{End}(\overline{T}_{\alpha,K})^{\text{op}}. \end{array}$$

□

The algebra homomorphism $\overline{\psi}_{\alpha,R}$ in Lemma 3.28 is an R -linear version of the homomorphism $\overline{\psi}_{\alpha,\mathbf{k}}$ in Lemma 3.21.

3.14 The proof of invertibility

The goal of this section is to prove that the homomorphism $\overline{\psi}_{\alpha,R}$ in Lemma 3.28 is an isomorphism.

Consider the functors

$$\begin{aligned} \Psi_{\alpha}^{\nu}: \mathbf{A}_R^{\nu}[\alpha] &\rightarrow \text{mod}(H_{\alpha,R}^{\nu}(q_e)), & M &\mapsto \text{Hom}(T_{\alpha,R}, M), \\ \overline{\Psi}_{\alpha}^{\nu}: \mathcal{A}_R^{\nu}[\alpha] &\rightarrow \text{mod}(H_{\alpha,R}^{\nu}(q_e)), & M &\mapsto \text{Hom}(\overline{T}_{\alpha,R}, M), \end{aligned}$$

where $\text{Hom}(T_{\alpha,R}, M)$ and $\text{Hom}(\overline{T}_{\alpha,R}, M)$ are considered as $H_{d,R}^{\nu}(q_e)$ -modules with respect to the homomorphisms $\psi_{\alpha,R}$ and $\overline{\psi}_{\alpha,R}$.

Let us abbreviate $\Psi = \Psi_{\alpha}^{\nu}$, $\overline{\Psi} = \overline{\Psi}_{\alpha}^{\nu}$, $T_R = T_{\alpha,R}$ and $\overline{T}_R = \overline{T}_{\alpha,R}$. We may write $\Psi_R, \overline{\Psi}_R$ to specify the ring R . For $\lambda \in \mathcal{P}_{\alpha,\nu}^l$ denote by $S[\lambda]_R$ the Specht module of $H_{\alpha,R}^{\nu}(q_e)$. We will use similar notation for \mathbf{k} or K instead of R . See also [19, Sec. 2.4.3].

Lemma 3.29. (a) *The homomorphism $\overline{\psi}_{\alpha,K}: H_{\alpha,K}^{\nu}(q_e) \rightarrow \text{End}(\overline{T}_K)^{\text{op}}$ is an isomorphism.*

(b) *For each $\lambda \in \mathcal{P}_{\alpha,\nu}^l$ we have $\overline{\Psi}(\overline{\Delta}[\lambda]_K) \simeq S[\lambda]_K$.*

Proof. First, we prove that the homomorphism $\overline{\psi}_{\alpha,K}$ is injective. The algebra $H_{\alpha,K}^{\nu}(q_e)$ is finite dimensional and semisimple. Its center is spanned by the idempotents $e(\tilde{\alpha})$ such that $\tilde{\alpha} \in Q_I^+$ and $\pi_e(\tilde{\alpha}) = \alpha$. The idempotent $e(\tilde{\alpha})$ acts on $T_K = F_{\alpha}(\overline{\Delta}[\emptyset]_K)$ by projection onto $F_{\tilde{\alpha}}(\overline{\Delta}[\emptyset]_K)$. Thus, to prove the injectivity of $\overline{\psi}_{\alpha,K}$ we need to check that $F_{\tilde{\alpha}}(\overline{\Delta}[\emptyset]_K)$ is nonzero whenever $e(\tilde{\alpha})$ is nonzero. Similarly to the argument in Section 3.7, we see that the equivalence $\theta: \mathbf{A}_K^{\nu} \simeq \mathcal{A}_K^{\nu}$ yields an isomorphism of Grothendieck groups $[\mathbf{A}_K^{\nu}] \simeq [\mathcal{A}_K^{\nu}]$ that commutes with functors F_j . Thus the module $F_{\tilde{\alpha}}(\overline{\Delta}[\emptyset]_K) \in \mathcal{A}_K^{\nu}$ is nonzero if and only if the module $F_{\tilde{\alpha}}(\Delta[\emptyset]_K) \in \mathbf{A}_K^{\nu}$ is nonzero. By [19, Prop. 5.22 (d)], the module $F_{\tilde{\alpha}}(\Delta[\emptyset]_K) \in \mathbf{A}_K^{\nu}$ is nonzero whenever $e(\tilde{\alpha})$ is nonzero. Thus $\overline{\psi}_{\alpha,K}$ is injective.

Thus it is also surjective because

$$\dim_K H_{\alpha,K}^\nu(q_e) = \dim_K \operatorname{End}(T_K)^{\operatorname{op}} = \dim_K \operatorname{End}(\overline{T}_K)^{\operatorname{op}},$$

where the first equality holds by [19, Prop. 5.22 (d)] and the second holds by (29). This implies part (a).

The discussion above implies that \overline{T}_K contains each $\overline{\Delta}[\lambda]_K$, $\lambda \in \mathcal{P}_\alpha^l$ as a direct factor. In particular \overline{T}_K is a projective generator of $\mathcal{A}_K^\nu[\alpha]$. Thus $\overline{\Psi}_K$ is an equivalence of categories. It must take $\overline{\Delta}[\lambda]_K$ to $S[\lambda]_K$ because $S[\lambda]_K$ is the unique simple module in the block $\operatorname{mod}(H_{\alpha,K}^\nu(q_e))$ of $\operatorname{mod}(H_{\alpha,K}^\nu(q_e))$. \square

Lemma 3.30. (a) *The homomorphism $\overline{\psi}_{\alpha,K}: H_{\alpha,R}^\nu(q_e) \rightarrow \operatorname{End}(\overline{T}_R)^{\operatorname{op}}$ is an isomorphism.*

(b) *For each $\lambda \in \mathcal{P}_{\alpha,\nu}^l$ we have $\overline{\Psi}(\overline{\Delta}[\lambda]_R) \simeq S[\lambda]_R$.*

Proof. Consider the endomorphism u of $H_{\alpha,R}^\nu(q_e)$ obtained from the following chain of homomorphisms

$$u: H_{\alpha,R}^\nu(q_e) \xrightarrow{\overline{\psi}_{\alpha,R}} \operatorname{End}_{\mathcal{A}^\nu}(\overline{T}_R)^{\operatorname{op}} \xrightarrow{\theta_\alpha^{-1}} \operatorname{End}_{\mathbf{A}^\nu}(T_R)^{\operatorname{op}} \xrightarrow{\psi_{\alpha,R}^{-1}} H_{\alpha,R}^\nu(q_e).$$

The invertibility of $\overline{\psi}_{\alpha,R}$ is equivalent to the invertibility of u . By [19, Prop. 2.23] to prove that u is an isomorphism it is enough to show that its localization $Ku: H_{\alpha,K}^\nu(q_e) \rightarrow H_{\alpha,K}^\nu(q_e)$ is an isomorphism and that Ku induces the identity map on Grothendieck groups $[\operatorname{mod}(H_{\alpha,K}^\nu(q_e))] \rightarrow [\operatorname{mod}(H_{\alpha,K}^\nu(q_e))]$. The bijectivity of Ku follows from Lemma 3.29 (a).

Now we check the condition on the Grothendieck group. We already know from [19, Prop. 5.22 (c)] and the proof of Lemma 3.29 that Ψ_K and $\overline{\Psi}_K$ are equivalences of categories. Thus, by semisimplicity of the categories $\mathbf{A}_K^\nu[\alpha]$, $\mathcal{A}_K^\nu[\alpha]$ and $\operatorname{mod}(H_{\alpha,K}^\nu(q_e))$, we have an isomorphism of functors $\Psi_K \simeq \overline{\Psi}_K \circ \theta_\alpha$ because $\Psi_K(M) \simeq \overline{\Psi}_K \circ \theta_\alpha(M)$ for each $M \in \mathbf{A}_K^\nu[\alpha]$. This implies that Ku is the identity on the Grothendieck group. This proves part (a).

Part (b) follows from Lemma 3.29 and the characterization of Specht modules, see [19, Sec. 2.4.3]. \square

Remark 3.31. There is no reason why the automorphism $u: H_{\alpha,R}^\nu(q_e) \rightarrow H_{\alpha,R}^\nu(q_e)$ in the proof of Lemma 3.30 should be identity. Because of this the functor Ψ has no reason to coincide with $\overline{\Psi} \circ \theta_\alpha$. However the automorphism u of $H_{\alpha,R}^\nu(q_e)$ induces an autoequivalence u^* of $\operatorname{mod}(H_{\alpha,R}^\nu(q_e))$ such that we have

$$\Psi = u^* \circ \overline{\Psi} \circ \theta_\alpha. \quad (35)$$

Now, specializing to \mathbf{k} , we obtain the following.

Corollary 3.32. (a) *The homomorphism $\overline{\psi}_{\alpha,\mathbf{k}}: R_{\alpha,\mathbf{k}}^\Lambda \rightarrow \operatorname{End}(\overline{T}_{\mathbf{k}})^{\operatorname{op}}$ is an isomorphism.*

(b) *For each $\lambda \in \mathcal{P}_{\alpha,\nu}^l$ we have $\overline{\Psi}_{\mathbf{k}}(\overline{\Delta}[\lambda]_{\mathbf{k}}) \simeq S[\lambda]_{\mathbf{k}}$.* \square

3.15 Rational Cherednik algebras

Let R be a local commutative \mathbb{C} -algebra with residue field \mathbb{C} . Let W be a complex reflection group. Denote by $S = S(W)$ and \mathcal{A} the set of pseudo-reflections in W and the set of reflection hyperplanes respectively. Let \mathfrak{h} be the reflection representation of W over R . Let $c: S \rightarrow R$ be a map which is constant on the W -conjugacy classes.

Denote by $\langle \bullet, \bullet \rangle$ the canonical pairing between \mathfrak{h}^* and \mathfrak{h} . For each $s \in S$ fix a generator $\alpha_s \in \mathfrak{h}^*$ of $\text{Im}(s|_{\mathfrak{h}^*} - 1)$ and a generator $\check{\alpha}_s \in \mathfrak{h}$ of $\text{Im}(s|_{\mathfrak{h}} - 1)$ such that $\langle \alpha_s, \check{\alpha}_s \rangle = 2$.

Definition 3.33. The *rational Cherednik algebra* $H_c(W, \mathfrak{h})_R$ is the quotient of the smash product $RW \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

$$[x, x'] = 0, \quad [y, y'] = 0, \quad [y, x] = \langle x, y \rangle - \sum_{s \in S} c_s \langle \alpha_s, y \rangle \langle x, \check{\alpha}_s \rangle s,$$

for each $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$. Here $T(\bullet)$ denotes the tensor algebra.

Denote by $\mathcal{O}_c(W, \mathfrak{h})_R$ the category \mathcal{O} of $H_c(W, \mathfrak{h})_R$, see [10, Sec. 3.2] and [19, Sec. 6.1.1]. Let E be an irreducible representation of $\mathbb{C}W$.

Definition 3.34. A *Verma module* associated with E is the following module in $\mathcal{O}_c(W, \mathfrak{h})_R$

$$\Delta_R(E) := \text{Ind}_{RW \ltimes \mathfrak{h}^*}^{H_c(W, \mathfrak{h})_R}(RE).$$

Here each element $x \in \mathfrak{h}^*$ acts on RE by zero.

The category $\mathcal{O}_c(W, \mathfrak{h})_R$ is a highest weight category over R with standard modules $\Delta_R(E)$.

We call a subgroup W' of W *parabolic* if it is a stabilizer of some point of $b \in \mathfrak{h}$. In this case W' is a complex reflection group with reflection representation $\mathfrak{h}/\mathfrak{h}^{W'}$, where $\mathfrak{h}^{W'}$ is the set of W' -stable points in \mathfrak{h} . Moreover, the map $c: S(W) \rightarrow R$ restricts to a map $c: S(W') \rightarrow R$. There are induction and restriction functors

$${}^{\mathcal{O}}\text{Ind}_{W'}^W: \mathcal{O}_c(W', \mathfrak{h}/\mathfrak{h}^{W'})_R \rightarrow \mathcal{O}_c(W, \mathfrak{h})_R, \quad {}^{\mathcal{O}}\text{Res}_{W'}^W: \mathcal{O}_c(W, \mathfrak{h})_R \rightarrow \mathcal{O}_c(W', \mathfrak{h}/\mathfrak{h}^{W'})_R,$$

see [2]. The definitions of these functors depend on b but their isomorphism classes are independent of the choice of b .

The following lemma holds.

Lemma 3.35. *Assume that W' and W'' are conjugated parabolic subgroups in W . Let $P \in \mathcal{O}_c(W, \mathfrak{h})_R$ be a projective module. Then the following conditions are equivalent*

- the module P is isomorphic to a direct factor of the module ${}^{\mathcal{O}}\text{Ind}_{W'}^W(P')$ for some projective module $P' \in \mathcal{O}_c(W', \mathfrak{h}/\mathfrak{h}^{W'})_R$,
- the module P is isomorphic to a direct factor of a module ${}^{\mathcal{O}}\text{Ind}_{W''}^W(P'')$ for some projective module $P'' \in \mathcal{O}_c(W'', \mathfrak{h}/\mathfrak{h}^{W''})_R$.

Proof. Let w be an element of W such that $wW'w^{-1} = W''$. The conjugation by w yields an isomorphism $W' \simeq W''$. Hence, the element w takes $\mathfrak{h}^{W'}$ to $\mathfrak{h}^{W''}$. Thus we get an algebra isomorphism $H_c(W', \mathfrak{h}/\mathfrak{h}^{W'})_R \simeq H_c(W'', \mathfrak{h}/\mathfrak{h}^{W''})_R$ and an equivalence of categories $\mathcal{O}_c(W', \mathfrak{h}/\mathfrak{h}^{W'})_R \simeq \mathcal{O}_c(W'', \mathfrak{h}/\mathfrak{h}^{W''})_R$. Moreover, the conjugation by w yields an automorphism t of $H_c(W, \mathfrak{h})_R$ such that for each $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$, $u \in W$ we have

$$t(x) = w(x), \quad t(y) = w(y), \quad t(u) = wuw^{-1}.$$

The following diagram of functors is commutative up to equivalence of functors

$$\begin{array}{ccc} \mathcal{O}_c(W, \mathfrak{h})_R & \xleftarrow{t^*} & \mathcal{O}_c(W, \mathfrak{h})_R \\ \circ_{\text{Ind}_{W'}}^{\uparrow} & & \circ_{\text{Ind}_{W''}}^{\uparrow} \\ \mathcal{O}_c(W', \mathfrak{h}/\mathfrak{h}^{W'})_R & \xleftarrow{\quad} & \mathcal{O}_c(W'', \mathfrak{h}/\mathfrak{h}^{W''})_R \end{array}$$

To conclude, we need only to prove that the pull-back t^* induces the identity map on the Grothendieck group of $\mathcal{O}_c(W, \mathfrak{h})_R$ (and thus it maps each projective module to an isomorphic one). This is true because t^* maps each Verma module $\Delta(E)_R$ to an isomorphic one because the representation E of W does not change the isomorphism class when we twist the W -action by an inner automorphism. \square

3.16 Cyclotomic rational Cherednik algebras

From now on, we assume that R is either a local analytic deformation ring in general position of dimension ≤ 2 or a field. Let \mathbf{k} and K be its residue field and the field of fractions respectively.

Let $\Gamma \simeq \mathbb{Z}/l\mathbb{Z}$ be the group of complex l th roots of unity and set $\Gamma_d = \mathfrak{S}_d \ltimes \Gamma^d$. For $\gamma \in \Gamma$, $r \in [1, l]$ denote by γ_r the element of Γ^d having γ at the position r and 1 at other positions. Let $s_{r,t}$ be the transposition in \mathfrak{S}_d exchanging r and t . For $\gamma \in \Gamma$, $r, t \in [1, l]$ set $s_{r,t}^\gamma := s_{r,t} \gamma_r \gamma_t^{-1} \in \Gamma_d$. From now on we suppose that the group W is Γ_d and $\mathfrak{h} = R^d$ is the obvious reflection representation of Γ_d . Assume also that h, h_1, \dots, h_{l-1} are some elements of R and set $h_{-1} = h_{l-1}$. Let us chose the parameter c in the following way

$$\begin{aligned} c(s_{r,t}^\gamma) &= -h \quad \text{for each } r, t \in [1, l], r \neq t, \gamma \in \Gamma, \\ c(\gamma_r) &= -\frac{1}{2} \sum_{p=0}^{l-1} \gamma^{-p} (h_p - h_{p-1}) \quad \text{for each } r \in [1, l], \gamma \in \Gamma, \gamma \neq 1. \end{aligned}$$

Let ν_1, \dots, ν_l be as above. We set

$$h = -1/\kappa, \quad h_p = -(\nu_{p+1} + \tau_{p+1})/\kappa - p/l, \quad p \in [1, l-1].$$

Let us abbreviate $\mathcal{O}_R^\nu[d] = \mathcal{O}_c(\Gamma_d, R^d)_R$. Consider the KZ-functor $\text{KZ}_d^\nu: \mathcal{O}_R^\nu[d] \rightarrow \text{mod}(H_{d,R}^\nu(q_e))$ introduced in [19, Sec. 6]. Denote by $^*\mathcal{O}_R^\nu[d]$ the category defined

in the same way as $\mathcal{O}_R^\nu[d]$ with replacement of (ν_1, \dots, ν_l) by $(-\nu_l, \dots, -\nu_1)$ and (τ_1, \dots, τ_l) by $(-\tau_l, \dots, -\tau_1)$. Similarly, denote by ${}^*H_{d,R}^\nu(q_e)$ the affine Hecke algebra defined in the same way as $H_{d,R}^\nu(q_e)$ with the replacement of parameters as above. There is also a KZ-functor ${}^*KZ_d^\nu: {}^*\mathcal{O}_R^\nu[d] \rightarrow \text{mod}({}^*H_{d,R}^\nu(q_e))$.

The simple $\mathbb{C}\Gamma_d$ -modules are labeled by the set \mathcal{P}_d^l . We write $E(\lambda)$ for the simple module corresponding to λ . Set $\Delta[\lambda]_R = \Delta(E(\lambda))_R$. Similarly, write $P[\lambda]_R$ and $T[\lambda]_R$ for the projective and tilting object in $\mathcal{O}_R^\nu[d]$ with index λ .

The category ${}^*\mathcal{O}_R^\nu[d]$ is the Ringel dual of the category $\mathcal{O}_R^\nu[d]$, see [19, Sec. 6.2.4]. In particular we have an equivalence between the categories of standardly filtered objects $\mathcal{R}: {}^*\mathcal{O}_R^\nu[d]^\Delta \rightarrow (\mathcal{O}_R^\nu[d]^\Delta)^{\text{op}}$. Let $\text{proj}(R)$ be the category of projective finitely generated R -modules. There is an algebra isomorphism

$$\iota: H_{d,R}^\nu(q_e) \rightarrow ({}^*H_{d,R}^\nu(q_e))^{\text{op}}, \quad T_r \mapsto -q_e T_r^{-1}, \quad X_r \mapsto X_r^{-1},$$

see [19, Sec. 6.2.4]. It induces an equivalence

$$\mathcal{R}_H = \iota^*(\bullet^\vee): \text{mod}({}^*H_{d,R}^\nu(q_e)) \cap \text{proj}(R) \rightarrow (\text{mod}(H_{d,R}^\nu(q_e)) \cap \text{proj}(R))^{\text{op}},$$

where \bullet^\vee is the dual as an R -module. By [19, (6.3)], the following diagram of functors is commutative

$$\begin{array}{ccc} {}^*\mathcal{O}_R^\nu[d]^\Delta & \xrightarrow{\mathcal{R}} & (\mathcal{O}_R^\nu[d]^\Delta)^{\text{op}} \\ {}^*KZ_d^\nu \downarrow & & KZ_d^\nu \downarrow \\ \text{mod}({}^*H_{d,R}^\nu(q_e)) \cap \text{proj}(R) & \xrightarrow{\mathcal{R}_H} & (\text{mod}(H_{d,R}^\nu(q_e)) \cap \text{proj}(R))^{\text{op}}. \end{array} \quad (36)$$

Lemma 3.36. *Assume that $d = 1$. For each l -partition of 1 λ we have an isomorphism of $H_{1,R}^\nu(q_e)$ -modules $KZ_1^\nu(P[\lambda]_R) \simeq \overline{\Psi}_1^\nu(\overline{T}[\lambda]_R)$.*

Proof. The proof is similar to the proof of [19, Prop. 6.7]. The commutativity of the diagram (36) implies

$$KZ_1^\nu(P[\lambda]_R) = KZ_1^\nu(\mathcal{R}(T[\lambda]_R)) = \mathcal{R}_H({}^*KZ_1^\nu(T[\lambda]_R)).$$

To conclude, we just need to compare the highest weight covers $\mathcal{R}_H \circ {}^*KZ_1^\nu$ and $\overline{\Psi}_1^\nu$ of $H_{1,R}^\nu(q_e)$ using Lemma 3.29 (b) and [19, Prop. 2.21]. \square

Let $O_{\mu,R}^+$ be the affine parabolic category \mathcal{O} associated with the parabolic type consisting of the single block of size $\nu_1 + \dots + \nu_l$. We define the categories $\mathbf{A}_R^+[d]$, $\mathcal{A}_R^+[d]$ and $O_R^+[d]$ similarly. In this case we will also write the upper index $+$ in the notation of modules and functors (for example $\Delta^+[\lambda]_R$, $T_{d,R}^+$, KZ_d^+ , etc.) Let also $H_{d,R}^+(q_e)$ be the cyclotomic Hecke algebra with $l = 1$. It is isomorphic to the Hecke algebra of \mathfrak{S}_d .

We can prove the following result.

Lemma 3.37. *For each $\lambda \in \mathcal{P}_2^1$ we have $KZ_2^+(P^+[\lambda]_R) \simeq \overline{\Psi}_2^+(\overline{T}^+[\lambda]_R)$.* \square

Proof. Similarly to the proof of Lemma 3.36 we compare the highest weight covers $\mathcal{R}_H \circ^* \text{KZ}_2^+$ and $\overline{\Psi}_2^+$ of $H_{2,R}^+(q_e)$ using Lemma 3.29 (b) and [19, Prop. 2.21]. \square

Denote by $\text{Ind}_{d,+}^{d,\nu}$ the induction functor with respect to the inclusion $H_{d,R}^+(q_e) \subset H_{d,R}^\nu(q_e)$. We will also need the following lemma.

Lemma 3.38. *Assume $\nu_r \geq 2$ for each $r \in [1, l]$. Assume also that $e > 2$. For each $\lambda \in \mathcal{P}_2^1$ there exists a tilting module $\overline{T}_{\lambda,R} \in \mathcal{A}_R^\nu[2]$ such that $\Psi_2^\nu(\overline{T}_{\lambda,R}) \simeq \text{Ind}_{2,+}^{2,\nu}(\overline{\Psi}_2^+(\overline{T}^+[\lambda]_R))$.*

Proof. Set $\lambda^+ = (2)$, $\lambda^- = (1, 1)$. We have $\zeta_e \neq -1$ because $e > 2$. In this case the algebra $H_{2,\mathbf{k}}^+(\zeta_e)$ is semisimple. The category $\mathcal{A}_{\mathbf{k}}^+[2]$ is also semisimple. This implies

$$\overline{T}_{2,R}^+ \simeq \overline{\Delta}[\lambda^+]_R \oplus \overline{\Delta}[\lambda^-]_R = \overline{T}[\lambda^+]_R \oplus \overline{T}[\lambda^-]_R.$$

By definition, we have $\overline{\Psi}_2^+(\overline{T}_{2,R}^+) \simeq H_{2,R}^+(q_e)$ and $\overline{\Psi}_2^\nu(\overline{T}_{2,R}) \simeq H_{2,R}^\nu(q_e)$. This implies

$$\overline{\Psi}_2^\nu(\overline{T}_{2,R}) \simeq \text{Ind}_{2,+}^{2,\nu}(\overline{\Psi}_2^+(\overline{T}_{2,R}^+)).$$

By the proof of [19, Prop. 6.8], the functor Ψ_2^ν takes indecomposable factors of $\overline{T}_{2,R}$ to indecomposable modules. Thus, by (35), the functor $\overline{\Psi}_2^\nu$ takes indecomposable factors of $\overline{T}_{2,R}$ to indecomposable modules. Thus there is a decomposition $\overline{T}_{2,R} = \overline{T}_{\lambda^+,R} \oplus \overline{T}_{\lambda^-,R}$ such that $\overline{T}_{\lambda^+,R}, \overline{T}_{\lambda^-,R}$ satisfy the required properties. \square

3.17 Proof of Theorem 1.4

In this section we finally give a proof of our main result.

A priori there is no reason to have the following isomorphism of functors $\Psi_\alpha^\nu \simeq \overline{\Psi}_\alpha^\nu \circ \theta_\alpha$. However, we can modify the equivalence θ_α to make this true.

For $d_1 < d_2$ we have an inclusion $H_{d_1,R}^\nu(q_e) \subset H_{d_2,R}^\nu(q_e)$. Let $\text{Ind}_{d_1}^{d_2}: \text{mod}(H_{d_1,R}^\nu(q_e)) \rightarrow \text{mod}(H_{d_2,R}^\nu(q_e))$ be the induction with respect to this inclusion. The following lemma can be proved similarly to [19, Lem. 5.41].

Lemma 3.39. *Assume that $\nu_r > d$ for each $r \in [1, l]$. Then the following diagram of functors is commutative.*

$$\begin{array}{ccc} \mathcal{A}_R^\nu[d] & \xrightarrow{F} & \mathcal{A}_R^\nu[d+1] \\ \overline{\Psi}_d^\nu \downarrow & & \overline{\Psi}_{d+1}^\nu \downarrow \\ \text{mod}(H_{d,R}^\nu(q_e)) & \xrightarrow{\text{Ind}_d^{d+1}} & \text{mod}(H_{d+1,R}^\nu(q_e)) \end{array}$$

\square

For a partition λ denote by λ^* the transposed partition. For an l -partition $\lambda = (\lambda_1, \dots, \lambda_l)$ set $\lambda^* = ((\lambda_l)^*, \dots, (\lambda_1)^*)$. There is an algebra isomorphism

$$\text{IM}: H_{d,R}^\nu(q_e) \rightarrow {}^*H_{d,R}^\nu(q_e), \quad T_r \mapsto -q_e T_r^{-1}, \quad X_r \mapsto X_r^{-1},$$

see [19, Sec. 6.2.4]. Let $\text{IM}^*: \text{mod}({}^*H_{d,R}^\nu(q_e)) \rightarrow \text{mod}(H_{d,R}^\nu(q_e))$ be the induced equivalence of categories. We have

$$\text{IM}^*(S[\lambda^*]_R) \simeq S[\lambda]_R. \quad (37)$$

The following proposition is proved in [19, Thm. 6.9].

Proposition 3.40. *Assume that $\nu_r \geq d$ for each $r \in [1, l]$. Then there is an equivalence of categories $\gamma_d: {}^*\mathcal{O}_R^\nu[d] \simeq \mathbf{A}_R^\nu[d]$ taking $\Delta[\lambda^*]_R$ to $\Delta[\lambda]_R$. Moreover, we have the following isomorphism of functors $\Psi_d^\nu \circ \gamma_d \simeq \text{IM}^* \circ {}^*\text{KZ}_d^\nu$.*

Now, we prove a similar statement for $\mathcal{A}_R^\nu[d]$. For each reflection hyperplane H of the complex reflection group Γ_d let $W_H \subset \Gamma_d$ be the pointwise stabilizer of H .

Proposition 3.41. *Assume that $\nu_r \geq d$ for each $r \in [1, l]$ and $e > 2$. There is an equivalence of categories $\overline{\gamma}_d: {}^*\mathcal{O}_R^\nu[d] \simeq \mathcal{A}_R^\nu[d]$, taking $\Delta[\lambda^*]_R$ to $\overline{\Delta}[\lambda]_R$. Moreover, we have the following isomorphism of functors $\overline{\Psi}_d^\nu \circ \overline{\gamma}_d \simeq \text{IM}^* \circ {}^*\text{KZ}_d^\nu$.*

Proof. The proof is similar to the proof of [19, Thm. 6.9]. We set $\mathcal{C} = {}^*\mathcal{O}_R^\nu[d]$, $\mathcal{C}' = \mathcal{A}_R^\nu[d]$. Consider the following functors

$$\begin{aligned} Y: \mathcal{C} &\rightarrow \text{mod}(H_{d,R}^\nu(q_e)), & Y &= \text{IM}^* \circ {}^*\text{KZ}_d^\nu, \\ Y': \mathcal{C}' &\rightarrow \text{mod}(H_{d,R}^\nu(q_e)), & Y' &= \overline{\Psi}_d^\nu. \end{aligned}$$

By [19, Prop. 2.20] it is enough to check the following four conditions.

- (1) We have $Y(\Delta[\lambda^*]_R) \simeq Y'(\Delta[\lambda]_R)$ and the bijection $\Delta[\lambda^*]_R \mapsto \Delta[\lambda]_R$ between the sets of standard objects in \mathcal{C} and \mathcal{C}' respects the highest weight orders.
- (2) The functor Y is fully faithful on \mathcal{C}^Δ and \mathcal{C}^∇ .
- (3) The functor Y' is fully faithful on \mathcal{C}'^Δ and \mathcal{C}'^∇ .
- (4) For each reflection hyperplane H of Γ_d and each projective module $P \in \mathcal{O}(W_H)_R$ we have

$$\text{KZ}_d^\nu({}^*\mathcal{O}\text{Ind}_{W_H}^{\Gamma_d} P) \in F'(\mathcal{C}'^{\text{tilt}}).$$

It is explained in the proof of [19, Thm. 6.9] that condition (4) announced here implies the fourth condition in [19, Prop. 2.20].

We have $Y(\Delta[\lambda^*]_R) \simeq Y'(\Delta[\lambda]_R)$ by Lemma 3.30 (b), [19, Lem. 6.6] and (37). The composition of the equivalence $\theta_d: \mathbf{A}_R^\nu[d] \rightarrow \mathcal{A}_R^\nu[d]$ with the equivalence $\gamma_d: {}^*\mathcal{O}_R^\nu[d] \simeq \mathbf{A}_R^\nu[d]$ yields an equivalence of highest weight categories $\mathcal{C} \simeq \mathcal{C}'$ that takes $\Delta[\lambda^*]_R$ to $\Delta[\lambda]_R$. This implies (1).

Condition (2) is already checked in [19, Sec. 6.3.2].

The functor Ψ_d^ν is fully faithful on $\mathbf{A}_R^{\nu,\Delta}[d]$ and $\mathbf{A}_R^{\nu,\nabla}[d]$ by [19, Thm. 5.37 (c)]. Thus the functor $\overline{\Psi}_d^\nu$ is fully faithful on $\mathcal{A}_R^{\nu,\Delta}[d]$ and $\mathcal{A}_R^{\nu,\nabla}[d]$ by (35). This implies (3).

Let us check condition (4). There are two possibilities for the hyperplane H .

- The hyperplane is $\text{Ker}(\gamma_r - 1)$ for $r \in [1, d]$. By Lemma 3.35, we can assume that $H = \text{Ker}(\gamma_1 - 1)$. By Lemma 3.36 there exists a tilting module $\overline{T} \in \mathcal{A}_R^\nu[1]$ such that $\text{KZ}_1^\nu(P) \simeq \overline{\Psi}_1^\nu(\overline{T})$. We get

$$\text{KZ}_d^\nu({}^{\mathcal{O}}\text{Ind}_{W_H}^{\Gamma_d} P) \simeq \text{Ind}_1^d(\text{KZ}_1^\nu(P)) \simeq \text{Ind}_1^d(\overline{\Psi}_1^\nu(\overline{T})) \simeq \overline{\Psi}_d^\nu(F^{d-1}(\overline{T})).$$

Here the first isomorphism follows from [19, (6.1)], the third isomorphism follows from Lemma 3.39.

- The hyperplane is $\text{Ker}(s_{r,t}^\gamma - 1)$ for $r, t \in [1, d]$, $\gamma \in \Gamma$. By Lemma 3.35, we can assume that $H = \text{Ker}(s_{1,2})$. By Lemma 3.37 there is a tilting module $\overline{T}^+ \in \mathcal{A}_R^+[2]$ such that $\text{KZ}_2^+(P) \simeq \overline{\Psi}_2^+(\overline{T}^+)$. By Lemma 3.38 there is a tilting module $\overline{T} \in \mathcal{A}_R^\nu[2]$ such that $\text{Ind}_{2,+}^{2,\nu}(\overline{\Psi}_2^+(\overline{T}^+)) \simeq \overline{\Psi}_2^\nu(\overline{T})$. Thus we get $\text{Ind}_{2,+}^{2,\nu}\text{KZ}_2^+(P) \simeq \overline{\Psi}_2^\nu(\overline{T})$.

We obtain

$$\text{KZ}_d^\nu({}^{\mathcal{O}}\text{Ind}_{W_H}^{\Gamma_d} P) \simeq \text{Ind}_{2,+}^{d,\nu}(\text{KZ}_2^+(P)) \simeq \text{Ind}_{2,+}^{d,\nu}(\overline{\Psi}_2^\nu(\overline{T})) \simeq \overline{\Psi}_d^\nu(F^{d-2}(\overline{T})).$$

Here the first isomorphism follows from [19, (6.1)], the third isomorphism follows from Lemma 3.39. \square

Now, composing the equivalences of categories in Propositions 3.40, 3.41 we obtain the following result.

Corollary 3.42. *Assume that $\nu_r \geq d$ for each $r \in [1, l]$ and $e > 2$. There is an equivalence of categories $\theta'_d: \mathbf{A}_R^\nu[d] \simeq \mathcal{A}_R^\nu[d]$ such that we have the following isomorphism of functors $\overline{\Psi}_d^\nu \circ \theta'_d \simeq \Psi_d^\nu$. \square*

For each $\alpha \in Q_I^+$ such that $|\alpha| = d$ the equivalence $\theta'_d: \mathbf{A}_R^\nu[d] \simeq \mathcal{A}_R^\nu[d]$ restricts to an equivalence $\theta'_\alpha: \mathbf{A}_R^\nu[\alpha] \simeq \mathcal{A}_R^\nu[\alpha]$ such that we have an isomorphism of functors $\overline{\Psi}_\alpha^\nu \circ \theta'_\alpha \simeq \Psi_\alpha^\nu$.

From now on we work over the field \mathbb{C} . Recall that we fixed an isomorphism $R_\alpha^\Lambda \simeq H_\alpha^\nu(\zeta_e)$. The following lemma can be proved by the method used in [19, Sec. 5.9].

Lemma 3.43. *Assume that $\nu_r > |\alpha|$ for each $r \in [1, l]$. The following diagrams are commutative modulo an isomorphism of functors.*

$$\begin{array}{ccc} \mathbf{A}^\nu[\alpha] & \xrightarrow{F_k} & \mathbf{A}^\nu[\alpha + \alpha_k] \\ \Psi_\alpha^\nu \downarrow & & \Psi_{\alpha + \alpha_k}^\nu \downarrow \\ \text{mod}(R_\alpha^\Lambda) & \xrightarrow{F_k^\Lambda} & \text{mod}(R_{\alpha + \alpha_k}^\Lambda) \end{array}$$

$$\begin{array}{ccc}
\mathcal{A}^\nu[\alpha] & \xrightarrow{F_k} & \mathcal{A}^\nu[\alpha + \alpha_k] \\
\bar{\Psi}_\alpha^\nu \downarrow & & \bar{\Psi}_{\alpha + \alpha_k}^\nu \downarrow \\
\text{mod}(R_\alpha^\Lambda) & \xrightarrow{F_k^\Lambda} & \text{mod}(R_{\alpha + \alpha_k}^\Lambda)
\end{array}$$

□

Now, Theorem 1.4 follows from the following one.

Theorem 3.44. *Assume that $\nu_r > |\alpha|$ for each $r \in [1, l]$ and $e > 2$. Then the following diagram is commutative*

$$\begin{array}{ccc}
\mathcal{A}^\nu[\alpha] & \xrightarrow{F_k} & \mathcal{A}^\nu[\alpha + \alpha_k] \\
\theta'_\alpha \uparrow & & \theta'_{\alpha + \alpha_k} \uparrow \\
\mathbf{A}^\nu[\alpha] & \xrightarrow{F_k} & \mathbf{A}^\nu[\alpha + \alpha_k].
\end{array}$$

Proof. The result follows from Corollary 3.42, Lemma 3.43 and an argument similar to [20, Lem. 2.4]. □

4 Graded lifts of the functors

4.1 Graded categories

For any noetherian ring A , let $\text{mod}(A)$ be the category of finitely generated left A -modules. For any noetherian \mathbb{Z} -graded ring $A = \bigoplus_{n \in \mathbb{Z}} A_n$, let $\text{grmod}(A)$ be the category of \mathbb{Z} -graded finitely generated left A -modules. The morphisms in $\text{grmod}(A)$ are the morphisms which are homogeneous of degree zero. For each $M \in \text{grmod}(A)$ and each $r \in \mathbb{Z}$ denote by M_r the homogeneous component of degree r in M . For $n \in \mathbb{Z}$ let $M\langle n \rangle$ be the n th shift of grading on M , i.e., we have $(M\langle n \rangle)_r = M_{r-n}$. For each \mathbb{Z} -graded finite dimensional \mathbb{C} -vector space V , let $\dim_q V \in \mathbb{N}[q, q^{-1}]$ be its graded dimension, i.e., $\dim_q V = \sum_{r \in \mathbb{Z}} (\dim V_r) q^r$.

The following lemma is proved in [3, Lem. 2.5.3].

Lemma 4.1. *Assume that M is an indecomposable A -module of finite length. Then, if M admits a graded lift, this lift is unique up to grading shift and isomorphism.* □

Definition 4.2. A \mathbb{Z} -category (or a *graded category*) is an additive category $\tilde{\mathcal{C}}$ with a fixed auto-equivalence $T: \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}}$. We call T the shift functor. For each $X \in \tilde{\mathcal{C}}$ and $n \in \mathbb{Z}$, we set $X\langle n \rangle = T^n(X)$. A functor of \mathbb{Z} -categories is a functor commuting with the shift functor.

For a graded noetherian ring A the category $\text{grmod}(A)$ is a \mathbb{Z} -category where T is the shift of grading, i.e., for $M = \bigoplus_{n \in \mathbb{Z}} M_n \in \text{grmod}(A)$, $k \in \mathbb{Z}$, we have $T(M)_k = M_{k-1}$.

Definition 4.3. Let \mathcal{C} be an abelian category. We say that an abelian \mathbb{Z} -category $\tilde{\mathcal{C}}$ is a *graded version* of \mathcal{C} if there exists a functor $F_{\mathcal{C}}: \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ and a graded noetherian ring A such that we have the following commutative diagram, where the horizontal arrows are equivalences of categories and the top horizontal arrow is a functor of \mathbb{Z} -categories

$$\begin{array}{ccc} \tilde{\mathcal{C}} & \longrightarrow & \text{grmod}(A) \\ F_{\mathcal{C}} \downarrow & & \text{forget} \downarrow \\ \mathcal{C} & \longrightarrow & \text{mod}(A). \end{array}$$

In the setup of Definition 4.3, we say that an object $\tilde{X} \in \tilde{\mathcal{C}}$ is a *graded lift* of an object $X \in \mathcal{C}$ if we have $F_{\mathcal{C}}(\tilde{X}) \simeq X$. For objects $X, Y \in \mathcal{C}$ with fixed graded lifts \tilde{X}, \tilde{Y} the \mathbb{Z} -module $\text{Hom}_{\mathcal{C}}(X, Y)$ admits a \mathbb{Z} -grading given by $\text{Hom}_{\mathcal{C}}(X, Y)_n = \text{Hom}_{\tilde{\mathcal{C}}}(\tilde{X}\langle n \rangle, \tilde{Y})$. In the sequel we will often denote the object X and its graded lift \tilde{X} by the same symbol.

Definition 4.4. For two abelian categories $\mathcal{C}_1, \mathcal{C}_2$ with graded versions $\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2$ we say that the functor of \mathbb{Z} -categories $\tilde{\Phi}: \tilde{\mathcal{C}}_1 \rightarrow \tilde{\mathcal{C}}_2$ is a *graded lift* of a functor $\Phi: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ if $F_{\mathcal{C}_2} \circ \tilde{\Phi} = \Phi \circ F_{\mathcal{C}_1}$.

4.2 The truncated category \mathcal{O}

We can extend the Bruhat order \leq on \widetilde{W} to an order \leq on \widehat{W} in the following way. For each $w_1, w_2 \in \widehat{W}$ we have $w_1 \leq w_2$ if and only if there exists $n \in \mathbb{Z}$ such that $w_1\pi^n, w_2\pi^n \in \widetilde{W}$ and we have $w_1\pi^n \leq w_2\pi^n$ in \widetilde{W} . Note that the order on \widehat{W} is defined in such a way that for $w_1, w_2 \in \widehat{W}$ we can have $w_1 \leq w_2$ only if $\widetilde{W}w_1 = \widetilde{W}w_2$.

Fix $\mu = (\mu_1, \dots, \mu_e) \in X_I[N]$. Let W_{μ} be the stabilizer of the weight $1_{\mu} \in P$ in \widetilde{W} (or equivalently in \widehat{W}). Let J_{μ} (resp. $J_{\mu,+}$) be the set of shortest (resp. longest) representatives of the cosets \widehat{W}/W_{μ} in \widehat{W} . For each $v \in \widehat{W}$ put ${}^vJ_{\mu} = \{w \in J_{\mu}; w \leq v\}$ and ${}^vJ_{\mu,+} = \{w \in J_{\mu,+}; w \leq v\}$. They are finite posets.

Assume that R is a local deformation ring. Let ${}^vO_{\mu,R}$ be the Serre subcategory of $O_{\mu,R}$ generated by the modules $\Delta_R^{w(1_{\mu})}$ with $w \in {}^vJ_{\mu}$. This is a highest weight category, see [21, Lem. 3.7]. Note that the definition of the category ${}^vO_{\mu,R}$ does not change if we replace v by the minimal length element in vW_{μ} (i.e., by an element of J_{μ}). However, in some situations it will be more convenient to assume that v is maximal in vW_{μ} (and not minimal).

Recall the decomposition

$$O_{\mu,R} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_{\pi^n(1_{\mu}),R}$$

in (25). Note that the definition of the order on \widehat{W} implies that the category ${}^vO_{\mu,R}$ lies in $\mathcal{O}_{\pi^n(1_{\mu}),R}$, where $n \in \mathbb{Z}$ is such that $v \in \widetilde{W}\pi^n$.

4.3 Linkage

We still consider the non-parabolic category O . In particular we have $l = N$.

Let \mathbf{k} be a deformation ring that is a field. Recall that the affine Weyl group \widetilde{W} is generated by reflections s_α , where α is a real affine root. Now we consider the following equivalence relation $\sim_{\mathbf{k}}$ on P . We define it as the equivalence relation generated by $\lambda_1 \sim_{\mathbf{k}} \lambda_2$ when $\tilde{\lambda}_1 + \hat{\rho} = s_\alpha(\tilde{\lambda}_2 + \hat{\rho})$ for some real affine root α . The definition of $\sim_{\mathbf{k}}$ depends on \mathbf{k} because the definitions of $\tilde{\lambda}$ and $\hat{\rho}$ depend on the elements $\tau_r, \kappa \in \mathbf{k}$.

Now, let R be a deformation ring that is a local ring with residue field \mathbf{k} . Then for $\lambda_1, \lambda_2 \in P$ we write $\lambda_1 \sim_R \lambda_2$ if and only if we have $\lambda_1 \sim_{\mathbf{k}} \lambda_2$. Note that the definition of the equivalence relation above is motivated by [7, Thm. 3.2].

In the particular case when R is a local deformation ring, the equivalence relation \sim_R coincides with the equivalence relation $\sim_{\mathbb{C}}$ because we have $\tau_r = 0$ and $\kappa = e$ in the residue field of R . The relation $\sim_{\mathbb{C}}$ can be easily described in terms of the e -action of \widetilde{W} on P , introduced in Section 3.2. We have $\lambda_1 \sim_{\mathbb{C}} \lambda_2$ if and only if the elements $\lambda_1 + \rho$ and $\lambda_2 + \rho$ of $P^{(e)}$ are in the same \widetilde{W} -orbit.

Remark 4.5. Let \mathbf{k} be as above.

(a) Assume that for each $r, t \in [1, l]$ such that $r \neq t$ we have $\tau_r - \tau_t \notin \mathbb{Z}$. In this case the equivalence relation $\sim_{\mathbf{k}}$ is the equality.

(b) Assume that we have $\tau_r - \tau_t \in \mathbb{Z}$ for a unique couple (r, t) as above. In this case each equivalence class with respect to $\sim_{\mathbf{k}}$ contains at most two elements.

(c) Let R be as local deformation ring in general position with the field of fractions K . By (a), the equivalence relation \sim_K is just the equality. Now, let \mathfrak{p} be a prime ideal of height 1 in R . In this case, each equivalence class with respect to $\sim_{R_{\mathfrak{p}}}$ contains at most two elements (this follows from [19, Prop. 5.22 (a)], (a) and (b)).

The relation \sim_R yields a decomposition of the category $O_{-e, R}$ in a direct sum of subcategories, see [7, Prop. 2.8]. More precisely, let Λ be an \sim_R -equivalence class in P . Let $\mathcal{O}_{\Lambda, R}$ be the Serre subcategory of $O_{-e, R}$ generated by $\Delta(\lambda)$ for $\lambda \in \Lambda$. Then we have

$$O_{\mu, R} = \bigoplus_{\Lambda \subset P[\mu] - \rho} \mathcal{O}_{\Lambda, R}. \quad (38)$$

For example, if R is a local deformation ring, then this decomposition coincides with (25). The following lemma explains what happens after the base change, see [7, Lem. 2.9, Cor. 2.10].

Lemma 4.6. *The R and T be deformation rings that are local and let $R \rightarrow T$ be a ring homomorphism.*

(a) *The equivalence relation \sim_T is finer than the relation \sim_R .*

(b) *Let Λ be an equivalence class with respect to \sim_R . Then $T \otimes_R \mathcal{O}_{\Lambda, R}$ is equal to $\bigoplus_{\Lambda'} \mathcal{O}_{\Lambda', T}$, where the sum is taken by all \sim_T -equivalence classes Λ' in Λ . \square*

Definition 4.7. We say that the category $\mathcal{O}_{\Lambda,R}$ is *generic* if Λ contains a unique element and *subgeneric* if it contains exactly two elements.

More details about the structure of generic and subgeneric categories can be found in [8, Sec. 3.1].

4.4 Centers

We assume that R is a deformation ring that is a local ring with the residue field \mathbf{k} and the field of fractions K . Recall that we have $l = N$ because we consider the non-parabolic category \mathcal{O} .

Let Λ be an equivalence class in P with respect to \sim_R . Consider the category $\mathcal{O}_{\Lambda,R}$ as in (38). There is a partial order \leq on Λ such that $\lambda_1 \leq \lambda_2$ when $\lambda_2 - \lambda_1$ is a sum of simple roots. There exists an element $\lambda \in \Lambda$ such that λ is minimal in Λ with respect to this order. Assume that Λ is finite.

Definition 4.8. The *antidominant projective module* in $\mathcal{O}_{\Lambda,R}$ is the projective cover in $\mathcal{O}_{\Lambda,R}$ of the simple module $L_R(\lambda)$, where λ is the minimal element in Λ . (The existence of the protective cover as above is explained in [7, Thm. 2.7].)

This notion has no sense if Λ is infinite. However we can consider the truncated version. Fix $v \in \widehat{W}$. We have a truncation of the decomposition (38):

$${}^v\mathcal{O}_{\mu,R} = \bigoplus_{\Lambda} {}^v\mathcal{O}_{\Lambda,R}, \quad (39)$$

where we put ${}^v\mathcal{O}_{\Lambda,R} = \mathcal{O}_{\Lambda,R} \cap {}^v\mathcal{O}_{\mu,R}$.

By [7, Thm. 2.7] each simple module in ${}^v\mathcal{O}_{\mu,R}$ has a projective cover. As above, we denote by λ the element of Λ that is minimal in Λ with respect to the order \leq .

Definition 4.9. The *antidominant projective module* in ${}^v\mathcal{O}_{\Lambda,R}$ is the projective cover in ${}^v\mathcal{O}_{\Lambda,R}$ of the simple module $L_R(\lambda)$.

From now on we assume that R is a local deformation ring in general position, see Section 2.9. Let \mathbf{k} and K be the residue field and the field of fractions of R respectively. We set $h_0 = \tau_l - \tau_1 - \kappa + e$ and $h_r = \tau_{r+1} - \tau_r$ for $r \in [1, l-1]$. We have $h_r \neq 0$ for each $r \in [0, l-1]$ because the ring is assumed to be in general position. Under the assumption on R , the decomposition (39) contains only one term. Let ${}^vP_R^\mu$ be the antidominant projective module in ${}^v\mathcal{O}_{\mu,R}$, i.e., ${}^vP_R^\mu$ is the projective cover of $L_R^{\pi^n(1_\mu)}$, where n is such that we have $\pi^n \leq v$.

Lemma 4.10. (a) *The module ${}^vP_R^\mu$ has a Δ -filtration such that each Verma module in the category ${}^v\mathcal{O}_{\mu,R}$ appears exactly once as a subquotient in this Δ -filtration.*

(b) *For each base change $R' \otimes_R \bullet$, where R' is a deformation ring that is local, the module $R' \otimes_R {}^vP_R^\mu$ splits into a direct sum of anti-dominant projective modules in the blocks of the category ${}^v\mathcal{O}_{\mu,R'}$.*

Proof. The first part in (a) holds by [8, Thm. 2 (2)] and the second part in (a) holds by the proof of [8, Lem. 4]. Finally, (b) follows from [8, Rem. 5]. \square

We will need the following lemma.

Lemma 4.11. *Let A be a ring. Then the center $Z(\text{mod}(A))$ of the category $\text{mod}(A)$ is isomorphic to the center $Z(A)$ of the ring A .*

Proof. There is an obvious injective homomorphism $Z(A) \rightarrow Z(\text{mod}(A))$. We need only to check that it is also surjective.

Let z be an element of the center of $\text{mod}(A)$. By definition, z consists of an endomorphism z_M of M for each $M \in \text{mod}(A)$ such that these endomorphisms commute with all morphisms between the modules in $\text{mod}(A)$. Then z_A is an endomorphism of the A -module A that commutes with each other endomorphism of the A -module A . Thus z_A is a multiplication by an element a in the center of A .

Now we claim that for each module $M \in \text{mod}(A)$ the endomorphism z_M is the multiplication by a . Fix $m \in M$. Let $\phi: A \rightarrow M$ be the morphism of A -modules sending 1 to m . We have $\phi \circ z_A = z_M \circ \phi$. Then

$$z_M(m) = z_M \circ \phi(1) = \phi \circ z_A(1) = \phi(a) = am.$$

This completes the proof. \square

Let $Z_{\mu,R}$ (resp. ${}^v Z_{\mu,R}$) be the center of the category $O_{\mu,R}$ (resp. ${}^v O_{\mu,R}$).

Proposition 4.12. *The evaluation homomorphism ${}^v Z_{\mu,R} \rightarrow \text{End}({}^v P_R^\mu)$ is an isomorphism.* \square

Proof. The statement is proved in [8, Lem. 5]. There are however some subtle points that we explain.

Firstly, the statement of [8, Lem. 5] announces the result for the non-truncated category \mathcal{O} . But in fact, the main point of the proof of [8, Lem. 5] is to show the statement first in the truncated case.

Secondly, [8, Lem. 5] assumes that the deformation ring R is the localization of the symmetric algebra $S(\hat{\mathfrak{h}})$ at the maximal ideal generated by $\hat{\mathfrak{h}}$. Let us sketch the argument of [8, Lem. 5] to show that it works well for our assumption on R .

Denote by $ev_R: {}^v Z_{\mu,R} \rightarrow \text{End}({}^v P_R^\mu)$ the homomorphism in the statement. Let $I(R)$ be the set of prime ideals of height 1 in R . We claim that we have

$${}^v Z_{\mu,R} = \bigcap_{\mathfrak{p} \in I(R)} {}^v Z_{\mu,R_{\mathfrak{p}}}, \quad (40)$$

where the intersection is taken inside of ${}^v Z_{\mu,K}$. Really, let ${}^v A_{\mu,R}$ be the endomorphism algebra of the minimal projective generator of ${}^v O_{\mu,R}$. We have an equivalence of categories ${}^v O_{\mu,R} \simeq \text{mod}({}^v A_{\mu,R})$. By Lemma 4.11 we have an algebra isomorphism ${}^v Z_{\mu,R} \simeq Z({}^v A_{\mu,R})$. The same is true if we replace R by $R_{\mathfrak{p}}$ or K . By [7, Prop. 2.4] we have ${}^v A_{\mu,R_{\mathfrak{p}}} \simeq R_{\mathfrak{p}} \otimes_R {}^v A_{\mu,R}$, ${}^v A_{\mu,K} \simeq K \otimes_R {}^v A_{\mu,R}$.

The algebra ${}^vA_{\mu,R}$ is free over R as an endomorphism algebra of a projective module in ${}^vO_{\mu,R}$. Thus we have ${}^vA_{\mu,R} = \bigcap_{\mathfrak{p} \in I(R)} {}^vA_{\mu,R_{\mathfrak{p}}}$, where the intersection is taken in ${}^vA_{\mu,K}$. If we intersect each term in the previous formula with ${}^vZ_{\mu,K} = Z({}^vA_{\mu,K})$, we get (40).

Similarly, we have

$$\text{End}({}^vP_R^\mu) = \bigcap_{\mathfrak{p} \in I(R)} \text{End}(R_{\mathfrak{p}} \otimes_R {}^vP_R^\mu)$$

inside of $\text{End}(K \otimes_R {}^vP_R^\mu)$.

To conclude, we only need to show that the evaluation homomorphisms

$$ev_{R_{\mathfrak{p}}}: {}^vZ_{\mu,R_{\mathfrak{p}}} \rightarrow \text{End}(R_{\mathfrak{p}} \otimes_R {}^vP_R^\mu), \quad ev_K: {}^vZ_{\mu,K} \rightarrow \text{End}(K \otimes_R {}^vP_R^\mu)$$

are isomorphisms for each $\mathfrak{p} \in I(R)$ and that the following diagram is commutative

$$\begin{array}{ccc} \text{End}(R_{\mathfrak{p}} \otimes_R {}^vP_R^\mu) & \longrightarrow & \text{End}(K \otimes_R {}^vP_R^\mu) \\ \uparrow ev_{R_{\mathfrak{p}}} & & \uparrow ev_K \\ {}^vZ_{\mu,R_{\mathfrak{p}}} & \longrightarrow & {}^vZ_{\mu,K} \end{array}$$

The commutativity of the diagram is obvious. Since R is in general position, the category ${}^vO_{\mu,K}$ is semisimple, see Remark 4.5. Moreover, for each $\mathfrak{p} \in I(R)$, the category ${}^vO_{\mu,R_{\mathfrak{p}}}$ is a direct sum of blocks with at most two Verma modules in each, see Remark 4.5. Similarly, by Lemma 4.10 (b) the localisation $R_{\mathfrak{p}} \otimes_R {}^vP_R^\mu$ of the antidominant projective module splits into a direct sum of antidominant projective modules. Now, the invertibility of $ev_{R_{\mathfrak{p}}}$ and ev_K follows from [8, Lem. 2].

□

We will need the following lemma.

Lemma 4.13. *Assume that \mathcal{C}_1 is an abelian category and \mathcal{C}_2 is an abelian subcategory of \mathcal{C}_1 . Let $i: \mathcal{C}_2 \rightarrow \mathcal{C}_1$ be the inclusion functor. For each object $M \in \mathcal{C}_1$ we assume that M has a maximal quotient that is in \mathcal{C}_2 and we denote this quotient by $\tau(M)$. Then we have the following.*

- (a) *The functor $\tau: \mathcal{C}_2 \rightarrow \mathcal{C}_1$ is left adjoint to i .*
- (b) *Let L be a simple object in \mathcal{C}_2 . Assume that L has a projective cover P in \mathcal{C}_1 . Then $\tau(P)$ is a projective cover of L in \mathcal{C}_2 .*

Proof. Take $M \in \mathcal{C}_1$ and $N \in \mathcal{C}_2$. For each homomorphism $f: M \rightarrow N$ we have $M/\text{Ker}f \simeq \text{Im}f \in \mathcal{C}_2$. Thus $\text{Ker}f$ must contain the kernel of $M \rightarrow \tau(M)$. This implies that each homomorphism $f: M \rightarrow N$ factors through $\tau(M)$. This proves (a).

Now, we prove (b). We have a projective cover $p: P \rightarrow L$ in \mathcal{C}_1 . First, we claim that the object $\tau(P)$ is projective in \mathcal{C}_2 . Really, by (a) the functors from \mathcal{C}_2 to the category of \mathbb{Z} -modules $\text{Hom}_{\mathcal{C}_2}(\tau(P), \bullet)$ and $\text{Hom}_{\mathcal{C}_1}(P, \bullet)$ are isomorphic. Thus the first of them should be exact because the second one is exact by

the projectivity of P . This shows that $\tau(P)$ is projective in \mathcal{C}_2 . Denote by \bar{p} the surjection $\bar{p}: \tau(P) \rightarrow L$ induced by $p: P \rightarrow L$. Let t be the surjection $t: P \rightarrow \tau(P)$. We have $p = \bar{p} \circ t$. Now we must prove that each proper submodule $K \subset \tau(P)$ is in $\ker \bar{p}$. Really, if this is not true for some K , then $p(t^{-1}(K))$ is nonzero. Then we have $p(t^{-1}(K)) = L$ because the module L is simple. Then by the definition of a projective cover we must have $t^{-1}(K) = P$. This is impossible because t is surjective and K is a proper submodule of $\tau(P)$. \square

Remark 4.14. Let A be a graded noetherian ring. Let $I \subset A$ be a homogeneous ideal. Put $\mathcal{C}_1 = \text{mod}(A)$, $\mathcal{C}_2 = \text{mod}(A/I)$, $\tilde{\mathcal{C}}_1 = \text{grmod}(A)$, $\tilde{\mathcal{C}}_2 = \text{grmod}(A/I)$. There is an obvious inclusion of categories $i: \mathcal{C}_2 \rightarrow \mathcal{C}_1$ and it has an obvious graded lift $\tilde{i}: \tilde{\mathcal{C}}_2 \rightarrow \tilde{\mathcal{C}}_1$. The left adjoint functor τ to i is defined by $\tau(M) = M/IM$ and the left adjoint functor $\tilde{\tau}$ to \tilde{i} is also defined by $\tilde{\tau}(M) = M/IM$. This implies that the functor $\tilde{\tau}$ is a graded lift of τ .

Recall that we denote by s_0, \dots, s_{N-1} the simple reflections in \widetilde{W} .

Proposition 4.15. *We have an isomorphism*

$$Z_{\mu,R} \simeq \left\{ (z_w) \in \prod_{w \in J_\mu} R; z_w \equiv z_{s_r w} \pmod{h_r} \ \forall r \in [0, N-1], w \in J_\mu \cap s_r J_\mu \right\} \quad (41)$$

which maps an element $z \in Z_{\mu,R}$ to the tuple $(z_w)_{w \in J_\mu}$ such that z acts on the Verma module $\Delta_R^{w(1_\mu)}$ by z_w .

Proof. The statement is proved in [7, Thm. 3.6] in the case where R is the localization of the symmetric algebra $S(\hat{\mathfrak{h}})$ at the maximal ideal generated by $\hat{\mathfrak{h}}$. Similarly to Proposition 4.12, the same proof works under our assumption on the ring R . \square

Proposition 4.15 has the following truncated version that can be proved in the same way using the approach of localizations at the prime ideals of height 1. (See, for example, the proof of Proposition 4.12). For each such localization the result becomes clear by [7, Coro. 3.5] and Remark 4.5.

Proposition 4.16. *We have an isomorphism*

$${}^v Z_{\mu,R} \simeq \left\{ (z_w) \in \prod_{w \in {}^v J_\mu} R; z_w \equiv z_{s_r w} \pmod{h_r} \ \forall r \in [0, N-1], w \in {}^v J_\mu \cap s_r {}^v J_\mu \right\} \quad (42)$$

which maps an element $z \in {}^v Z_{\mu,R}$ to the tuple $(z_w)_{w \in {}^v J_\mu}$ such that z acts on the Verma module $\Delta_R^{w(1_\mu)}$ by z_w . \square

For each $v \in \widehat{W}$, set ${}^v J = \{w \in \widehat{W}; w \leq v\}$ and

$${}^v Z_R \simeq \left\{ (z_w) \in \prod_{w \in {}^v J} R; z_w \equiv z_{s_r w} \pmod{h_r} \forall r \in [0, N-1], w \in {}^v J \cap s_r {}^v J \right\}.$$

If v is in $J_{\mu,+}$, then the group W_μ acts on ${}^v Z_R$ by $w(z) = z'$ where $z'_x = z_{xw^{-1}}$ for each $x \in {}^v J$. Note that the algebra ${}^v Z_R^{W_\mu}$ of W_μ -invariant elements in ${}^v Z_R$ is obviously isomorphic to the right hand side in (42). Thus Proposition 4.16 identifies the center ${}^v Z_{\mu,R}$ of ${}^v O_{\mu,R}$ with ${}^v Z_R^{W_\mu}$.

4.5 The action on standard and projective modules

As above, we fix $k \in [0, e-1]$ and set $\mu' = \mu - \alpha_k$. From now on, we assume that R is an analytic local deformation ring in general position of dimension ≤ 2 with residue field $\mathbf{k} = \mathbb{C}$, see Section 2.9.

From now on we always assume that we have

$$W_\mu \subset W_{\mu'}. \quad (43)$$

This happens if and only if we have $\mu_k = 1$. In this case we have $J_{\mu'} \subset J_\mu$ and $J_{\mu',+} \subset J_{\mu,+}$. From now on we always assume that the element v is in $J_{\mu',+}$ (thus v is also in $J_{\mu,+}$). We have an inclusion of algebras ${}^v Z_{\mu',R} \subset {}^v Z_{\mu,R}$ because ${}^v Z_{\mu',R} \simeq {}^v Z_R^{W_{\mu'}}$ and ${}^v Z_{\mu,R} \simeq {}^v Z_R^{W_\mu}$. Let $\text{Res}: \text{mod}({}^v Z_{\mu,R}) \rightarrow \text{mod}({}^v Z_{\mu',R})$ and $\text{Ind}: \text{mod}({}^v Z_{\mu',R}) \rightarrow \text{mod}({}^v Z_{\mu,R})$ be the restriction and the induction functors. We may write $\text{Res}_\mu^{\mu'}$ and $\text{Ind}_\mu^{\mu'}$ to specify the parameters.

It is easy to see on Verma modules using two lemmas below that the functors E_k and F_k restrict to functors of truncated categories

$$F_k: {}^v O_{\mu,R}^\Delta \rightarrow {}^v O_{\mu',R}^\Delta, \quad E_k: {}^v O_{\mu',R}^\Delta \rightarrow {}^v O_{\mu,R}^\Delta.$$

Lemma 4.17. *For each $w \in \widehat{W}$, we have $F_k(\Delta_R^{w(1_\mu)}) \simeq \Delta_R^{w(1_{\mu'})}$.*

Proof. Since $\mu_k = 1$, the weight $w(1_\mu) \in P$ has a unique coordinate containing an element congruent to k modulo e . Let $r \in [1, N]$ be the position number of this coordinate. By Proposition 3.10 (e), we have $[F_k(\Delta_R^{w(1_\mu)})] = [\Delta_R^{w(1_\mu) + \epsilon_r}]$. The equality of classes in the Grothendieck group implies that we have an isomorphism of modules $F_k(\Delta_R^{w(1_\mu)}) \simeq \Delta_R^{w(1_\mu) + \epsilon_r}$. Finally, since $w(1_\mu) + \epsilon_r = w(1_{\mu'})$, we get $F_k(\Delta_R^{w(1_\mu)}) \simeq \Delta_R^{w(1_{\mu'})}$. \square

Lemma 4.18. *For each $w \in \widehat{W}$, we have $[E_k(\Delta_R^{w(1_{\mu'})})] = \sum_{z \in W_{\mu'}/W_\mu} [\Delta_R^{wz(1_\mu)}]$.*

Proof. By Proposition 3.10 (e), we have

$$[E_k(\Delta_R^{w(1_{\mu'})})] = \sum_r [\Delta_R^{w(1_{\mu'}) - \epsilon_r}], \quad (44)$$

where the sum is taken by all indices $r \in [1, N]$ such that the r th coordinate of $w(1_\mu)$ is congruent to $k+1$ modulo e . For each such r we have $w(1_{\mu'}) - \epsilon_r = wz(1_\mu)$ for a unique element $z \in W_{\mu'}/W_\mu$. Moreover, the obtained map $r \mapsto z$ is a bijection from the set of possible indices r to $W_{\mu'}/W_\mu$. Thus (44) can be rewritten as

$$[E_k(\Delta_R^{w(1_{\mu'})})] = \sum_{z \in W_{\mu'}/W_\mu} [\Delta_R^{wz(1_\mu)}].$$

□

Lemma 4.19. *We have $E_k({}^v P_R^{\mu'}) \simeq {}^v P_R^\mu$.*

Proof. By Lemma 4.10 (a), the class $[{}^v P_R^\mu]$ of ${}^v P_R^\mu$ in the Grothendieck group of ${}^v O_{\mu,R}^\Delta$ is the sum of all classes of Verma modules of the category ${}^v O_{\mu,R}^\Delta$ and similarly for $[{}^v P_R^{\mu'}]$. Taking the sum in the equality in Lemma 4.18 over all $w \in {}^v J_{\mu'}$, we get $[E_k({}^v P_R^{\mu'})] = [{}^v P_R^\mu]$. Finally, this yields an isomorphism $E_k({}^v P_R^{\mu'}) \simeq {}^v P_R^\mu$ because the modules $E_k({}^v P_R^{\mu'})$ and ${}^v P_R^\mu$ are projective. □

Fix an isomorphism $E_k({}^v P_R^{\mu'}) \simeq {}^v P_R^\mu$ as above. Then by functoriality it yields an algebra homomorphism $\text{End}({}^v P_R^{\mu'}) \rightarrow \text{End}({}^v P_R^\mu)$.

Lemma 4.20. *The following diagram of algebra homomorphisms is commutative*

$$\begin{array}{ccc} \text{End}({}^v P_R^{\mu'}) & \longrightarrow & \text{End}({}^v P_R^\mu) \\ \uparrow & & \uparrow \\ {}^v Z_{\mu',R} & \longrightarrow & {}^v Z_{\mu,R}, \end{array}$$

where the top horizontal map is as above, the bottom horizontal map is the inclusion and the vertical maps are the isomorphisms from Proposition 4.12.

Proof. Note that each element in $\text{End}({}^v P_R^\mu)$ is induced by the center ${}^v Z_{\mu,R}$. In particular, each endomorphism of ${}^v P_R^\mu$ preserves each submodule of ${}^v P_R^\mu$. Moreover, by Lemma 4.10 (a), each Verma module in ${}^v O_{\mu,R}$ is isomorphic to a subquotient of ${}^v P_R^\mu$. Thus, by Proposition 4.12 and Proposition 4.16, an element of $\text{End}({}^v P_R^\mu)$ is determined by its action on the subquotients of a Δ -filtration of ${}^v P_R^\mu$.

Fix an element $z = (z_w)$ in ${}^v Z_{\mu',R}$, see Proposition 4.16. Fix also a Δ -filtration of ${}^v P_R^{\mu'}$. The element z acts on ${}^v P_R^{\mu'}$ in such a way that it preserves each component of the Δ -filtration and the induced action on the subquotient $\Delta_R^{w(1_{\mu'})}$ of ${}^v P_R^{\mu'}$ is the multiplication by z_w .

For each $w \in \widehat{W}$, the module $E_k(\Delta_R^{w(1_{\mu'})})$ is Δ -filtered. The subquotients in this Δ -filtration can be described by Lemma 4.18. Since the functor E_k is exact, the Δ -filtration of ${}^v P_R^{\mu'}$ induces a Δ -filtration of ${}^v P_R^\mu \simeq E_k({}^v P_R^{\mu'})$. Thus the image of z by

$${}^v Z_{\mu',R} \rightarrow \text{End}({}^v P_R^{\mu'}) \rightarrow \text{End}({}^v P_R^\mu)$$

acts on the subquotients of the Δ -filtration of ${}^v P_R^\mu$ in the following way: it acts on the subquotient $\Delta_R^{w(1_\mu)}$ of ${}^v P_R^\mu$ by the multiplication by z_w . On the other hand, the image of z by

$${}^v Z_{\mu', R} \rightarrow {}^v Z_{\mu, R} \rightarrow \text{End}({}^v P_R^\mu)$$

acts on the subquotients in the same way. This proves the statement because an element of $\text{End}({}^v P_R^\mu)$ is determined by its action on the subquotients of a Δ -filtration of ${}^v P_R^\mu$. \square

4.6 The functor \mathbb{V}

Now, we assume that v is an arbitrary elements of \widehat{W} . We have a functor

$$\mathbb{V}_{\mu, R}: {}^v O_{\mu, R} \rightarrow \text{mod}({}^v Z_{\mu, R}), \quad M \mapsto \text{Hom}({}^v P_R^\mu, M).$$

Set ${}^v Z_\mu = \mathbb{C} \otimes_R {}^v Z_{\mu, R}$ and ${}^v Z = \mathbb{C} \otimes_R {}^v Z_R$. By [7, Prop. 2.6] we have $\mathbb{C} \otimes_R {}^v P_R^\mu = {}^v P^\mu$. Next, [7, Prop. 2.7] yields an algebra isomorphism ${}^v Z_\mu \simeq \text{End}({}^v P_\mu)$. Now, consider the functor

$$\mathbb{V}_\mu: {}^v O_\mu \rightarrow \text{mod}({}^v Z_\mu), \quad M \mapsto \text{Hom}({}^v P^\mu, M).$$

A Koszul grading on the category ${}^v O_\mu$ is constructed in [21]. Let us denote by ${}^v \widetilde{O}_\mu$ the graded version of this category.

The functor \mathbb{V} above has following properties.

- Proposition 4.21.** (a) *The functor $\mathbb{V}_{\mu, R}$ is fully faithful on ${}^v O_{\mu, R}^\Delta$.*
(b) *The functor \mathbb{V}_μ is fully faithful on projective objects in O_μ .*
(c) *The functor \mathbb{V}_μ admits a graded lift $\widetilde{\mathbb{V}}_\mu: {}^v \widetilde{O}_\mu \rightarrow \text{grmod}({}^v Z_\mu)$.*

Proof. Part (a) is [8, Prop. 2] (1). Part (b) is [21, Prop. 4.50] (b). Part (c) is given in the proof of [21, Lem. 5.10]. \square

4.7 The cohomology of Schubert varieties

All cohomology groups in this section have coefficients in \mathbb{C} .

Set $G = GL_N$. Let $B \subset G(\mathbb{C}((t)))$ be the standard Borel subgroup. Let $P_\mu \subset G(\mathbb{C}((t)))$ be the parabolic subgroup with Lie algebra $\widehat{\mathfrak{p}}_\mu$. Let X_μ be the partial affine flag ind-scheme $G(\mathbb{C}((t)))/P_\mu$. The affine Bruhat cells in X_μ are indexed by J_μ . For $w \in J_\mu$ we denote by $X_{\mu, w}$ (resp. $\overline{X}_{\mu, w}$) the corresponding finite dimensional affine Bruhat cell (resp. Schubert variety). Note that we have $X_{\mu, w} \simeq \mathbb{C}^{\ell(w)}$. The following statement is proved in [21, Prop. 4.43 (a)].

Lemma 4.22. *Assume $v \in J_\mu \cap \widetilde{W}$. There is an isomorphism of graded algebras between ${}^v Z_\mu$ and the cohomology $H^*(\overline{X}_{\mu, v})$.* \square

Now we are going to extend the notions $X_{\mu,w}$ and $\overline{X}_{\mu,w}$ to an arbitrary $w \in J_\mu$ in order to get an extended version of the previous lemma.

Let π be the cyclic shift of Dynkin diagram of type $A_{N-1}^{(1)}$ that takes the root α_i to the root α_{i-1} for $i \in \mathbb{Z}/N\mathbb{Z}$. It yields an automorphism $\pi: G \rightarrow G$. Then for each $n \in \mathbb{Z}$ we have a parabolic subgroup $\pi^n(P_\mu) \subset G(\mathbb{C}((t)))$. Recall that the symbol π also denotes an element of \widetilde{W} , see Section 3.2. Let X_μ^n be the partial affine flag ind-scheme defined in the same way as X_μ with respect to the parabolic subgroup $\pi^n(P_\mu) \subset G(\mathbb{C}((t)))$ instead of P_μ . In particular, we have $X_\mu = X_\mu^0$. Let us use the abbreviation $\pi^n(W_\mu)$ for the subgroup $\pi^n W_\mu \pi^{-n}$ of \widetilde{W} . Note that the group $\pi^n(W_\mu)$ is the Weyl group of the Levi of $\pi^n(P_\mu)$. The Bruhat cells and the Schubert varieties in X_μ^n are indexes by the shortest representatives of the cosets in $\widetilde{W}/\pi^n(W_\mu)$. For such a representative w let $X_{\mu,w}^n$ (resp. $\overline{X}_{\mu,w}^n$) be the Bruhat cell (resp. Schubert variety) in X_μ^n .

Assume that $v \in J_\mu$. Then v has a unique decomposition of the form $v = w\pi^n$, such that w is minimal in $w\pi^n(W_\mu)$. Then we set $X_{\mu,v} = X_{\mu,w}^n$ and $\overline{X}_{\mu,v} = \overline{X}_{\mu,w}^n$. Note that for $v \in J_\mu$ we have $X_{\mu,v} \simeq \mathbb{C}^{\ell(v)}$. We get the following generalization of the lemma above.

Lemma 4.23. *Assume $v \in J_\mu$. There is an isomorphism of graded algebras between ${}^vZ_\mu$ and the cohomology $H^*(\overline{X}_{\mu,v})$.*

Proof. Consider the decomposition $v = w\pi^n$ as above. By definition, the truncated category ${}^vO_\mu$ is a Serre subcategory of $\mathcal{O}_{\pi^n(1_\mu)}$. It is generated by modules $L^{x\pi^n(1_\mu)}$, where $x \in \widetilde{W}$ is such that $x \leq w$. Note also that the stabilizer of the weight $\pi^n(1_\mu)$ in \widetilde{W} is $\pi^n(W_\mu)$. Then, by [21, Prop. 4.43 (a)], we have an isomorphism of graded algebras ${}^vZ_\mu = H^*(\overline{X}_{\mu,w}^n)$. On the other hand the variety $\overline{X}_{\mu,v}$ is defined as $\overline{X}_{\mu,w}^n$. \square

Now, assume that $v \in J_{\mu',+}$. Recall that in this case we have an inclusion of algebras ${}^vZ_{\mu',R} \subset {}^vZ_{\mu,R}$ because of the assumption $W_\mu \subset W_{\mu'}$. We want to show that after the base change we get an inclusion of algebras ${}^vZ_{\mu'} \subset {}^vZ_\mu$. However, this is not obvious because the functor $\mathbb{C} \otimes_R \bullet$ is not left exact. But this fact can be justified using geometry. The injectivity of the homomorphism ${}^vZ_{\mu'} \rightarrow {}^vZ_\mu$ is a consequence of Lemma 4.24 below.

Denote by w_μ the longest elements in W_μ . The shortest elements in vW_μ and $vW_{\mu'}$ are respectively vw_μ and $vw_{\mu'}$. By Lemma 4.23, we have algebra isomorphisms ${}^vZ_\mu \simeq H^*(\overline{X}_{\mu,vw_\mu})$ and ${}^vZ_{\mu'} \simeq H^*(\overline{X}_{\mu',vw_{\mu'}})$.

The group \widetilde{W} is a Coxeter group. In particular we have a length function $\ell: \widetilde{W} \rightarrow \mathbb{N}$. We can extend it to \widetilde{W} by setting $\ell(w\pi^n) = \ell(w)$ for each $n \in \mathbb{Z}$ and $w \in \widetilde{W}$. Now we are ready to prove the following result.

Lemma 4.24. *There is the following isomorphism of graded ${}^vZ_{\mu'}$ -modules*

$${}^vZ_\mu \simeq \bigoplus_{r=0}^{\mu_{k+1}} {}^vZ_{\mu'}\langle 2r \rangle.$$

Proof. Let $J_{\mu'}^{\mu}$ be the set of shortest representatives of classes in $W_{\mu'}/W_{\mu}$. We have the following decomposition into affine cells

$$\overline{X}_{\mu, vw_{\mu}} = \coprod_{w \in {}^v J_{\mu}} X_{\mu, w} = \coprod_{w \in {}^v J_{\mu'}} \coprod_{x \in J_{\mu'}^{\mu}} X_{\mu, wx}.$$

This yields

$$\begin{aligned} {}^v Z_{\mu} &\simeq H^*(\overline{X}_{\mu, vw_{\mu}}) \\ &\simeq \bigoplus_{w \in {}^v J_{\mu}} H^*(X_{\mu, w}) \langle 2\ell(vw_{\mu}) - 2\ell(w) \rangle \\ &\simeq \bigoplus_{w \in {}^v J_{\mu'}} \bigoplus_{x \in J_{\mu'}^{\mu}} H^*(X_{\mu', wx}) \langle 2\ell(vw_{\mu}) - 2\ell(w) - 2\ell(x) \rangle. \end{aligned}$$

We also have $\overline{X}_{\mu, v} = \coprod_{w \in {}^v J_{\mu}} X_{\mu, w}$. This implies

$$\begin{aligned} {}^v Z_{\mu'} &\simeq H^*(\overline{X}_{\mu', vw_{\mu'}}) \\ &\simeq \bigoplus_{w \in {}^v J_{\mu'}} H^*(X_{\mu', w}) \langle 2\ell(vw_{\mu'}) - 2\ell(w) \rangle \end{aligned}$$

Note that we have $\ell(w_{\mu'}) - \ell(w_{\mu}) = \mu_{k+1}$. Moreover, for each $w \in {}^v J_{\mu'}$ and $x \in J_{\mu'}^{\mu}$ the variety $X_{\mu, wx}$ is an affine fibration over $X_{\mu', w}$. This implies

$${}^v Z_{\mu} \simeq \bigoplus_{x \in J_{\mu'}^{\mu}} {}^v Z_{\mu'} \langle 2\ell(vw_{\mu}) - 2\ell(vw_{\mu'}) - 2\ell(x) \rangle = \bigoplus_{r=0}^{\mu_{k+1}} {}^v Z_{\mu'} \langle 2r \rangle.$$

□

We will write Ind and Res for the induction and restriction functors

$$\text{Ind}: \text{mod}({}^v Z_{\mu'}) \rightarrow \text{mod}({}^v Z_{\mu}), \quad \text{Res}: \text{mod}({}^v Z_{\mu}) \rightarrow \text{mod}({}^v Z_{\mu'}).$$

We fix the graded lifts of $\overline{\text{Res}}$ and $\widetilde{\text{Ind}}$ of the functors Res and Ind in the following way

$$\widetilde{\text{Res}}(M) = M \langle -\mu_{k+1} \rangle, \quad \widetilde{\text{Ind}}(M) = {}^v Z_{\mu} \otimes_{{}^v Z_{\mu'}} M.$$

Now, Lemma 4.24 implies the following.

Corollary 4.25. (a) *The pair of functors (Res, Ind) is biadjoint.*

(b) *The pairs of functors $(\widetilde{\text{Ind}}, \widetilde{\text{Res}} \langle \mu_{k+1} \rangle)$ and $(\widetilde{\text{Res}}, \widetilde{\text{Ind}} \langle -\mu_{k+1} \rangle)$ are adjoint.*

(c)

$$\widetilde{\text{Res}} \circ \widetilde{\text{Ind}} = \text{Id}^{\oplus [\mu_{k+1}+1]_q} := \bigoplus_{r=0}^{\mu_{k+1}} \text{Id} \langle 2r - \mu_{k+1} \rangle,$$

where Id is the identity endofunctor of the category $\text{grmod}(Z_{\mu'})$. □

4.8 Graded lifts of the functors

As above we assume $W_\mu \subset W_{\mu'}$ and that $v \in J_{\mu',+}$.

Lemma 4.26. *The following diagram of functors is commutative*

$$\begin{array}{ccc} {}^vO_{\mu,R}^\Delta & \xrightarrow{F_k} & {}^vO_{\mu',R}^\Delta \\ \mathbb{V}_{\mu,R} \downarrow & & \mathbb{V}_{\mu',R} \downarrow \\ \text{mod}({}^vZ_{\mu,R}) & \xrightarrow{\text{Res}} & \text{mod}({}^vZ_{\mu',R}). \end{array}$$

Proof. Let M be an object in ${}^vO_{\mu,R}^\Delta$. We have the following chain of isomorphisms of ${}^vZ_{\mu',R}$ -modules.

$$\begin{aligned} \mathbb{V}_{\mu',R} \circ F_k(M) &\simeq \text{Hom}({}^vP_R^{\mu'}, F_k(M)) \\ &\simeq \text{Hom}(E_k({}^vP_R^{\mu'}), M) \\ &\simeq \text{Hom}({}^vP_R^\mu, M) \\ &\simeq \mathbb{V}_{\mu,R}(M) \end{aligned}$$

Here, the ${}^vZ_{\mu,R}$ -modules in the last two lines are considered as ${}^vZ_{\mu',R}$ -modules with respect to the inclusion ${}^vZ_{\mu',R} \subset {}^vZ_{\mu,R}$. The third isomorphism in the chain is an isomorphism of ${}^vZ_{\mu',R}$ -modules by Lemma 4.20. \square

Lemma 4.27. *The following diagram of functors is commutative*

$$\begin{array}{ccc} {}^vO_{\mu,R}^\Delta & \xleftarrow{E_k} & {}^vO_{\mu',R}^\Delta \\ \mathbb{V}_{\mu,R} \downarrow & & \mathbb{V}_{\mu',R} \downarrow \\ \text{mod}({}^vZ_{\mu,R}) & \xleftarrow{\text{Ind}} & \text{mod}({}^vZ_{\mu',R}). \end{array}$$

Proof. Let M be an object in ${}^vO_{\mu',R}^\Delta$. We have the following chain of isomorphisms of ${}^vZ_{\mu,R}$ -modules.

$$\begin{aligned} \mathbb{V}_{\mu,R} \circ E_k(M) &\simeq \text{Hom}({}^vP_R^\mu, E_k(M)) \\ &\simeq \text{Hom}(F_k({}^vP_R^\mu), M) \\ &\simeq \text{Hom}(\mathbb{V}_{\mu',R} \circ F_k({}^vP_R^\mu), \mathbb{V}_{\mu',R}(M)) \\ &\simeq \text{Hom}(\text{Res} \circ \mathbb{V}_{\mu,R}({}^vP_R^\mu), \mathbb{V}_{\mu',R}(M)) \\ &\simeq \text{Hom}({}^vZ_{\mu,R}, \mathbb{V}_{\mu',R}(M)) \\ &\simeq \text{Ind} \circ \mathbb{V}_{\mu',R}(M) \end{aligned}$$

Here the third isomorphism holds by Proposition 4.21 (a), the fourth isomorphism holds by Lemma 4.26. The last isomorphism holds because, by Corollary 4.25 (a), the functor $\text{Hom}_{{}^vZ_{\mu',R}}({}^vZ_{\mu,R}, \bullet)$, which is obviously right adjoint to Res , is isomorphic to Ind . \square

Now, Lemmas 4.26-4.27 imply the following.

Corollary 4.28. *The following diagrams of functors are commutative*

$$\begin{array}{ccc}
{}^v O_\mu & \xrightarrow{F_k} & {}^v O_{\mu'} \\
\mathbb{V}_\mu \downarrow & & \mathbb{V}_{\mu'} \downarrow \\
\text{mod}({}^v Z_\mu) & \xrightarrow{\text{Res}} & \text{mod}({}^v Z_{\mu'}), \\
\\
{}^v O_\mu & \xleftarrow{E_k} & {}^v O_{\mu'} \\
\mathbb{V}_\mu \downarrow & & \mathbb{V}_{\mu'} \downarrow \\
\text{mod}({}^v Z_\mu) & \xleftarrow{\text{Ind}} & \text{mod}({}^v Z_{\mu'}).
\end{array}$$

Proof. Passage to the residue field in Lemma 4.27 implies that the diagrams in the statement are commutative on Δ -filtered objects. A standard argument (see for example the proof of Lemma 4.29) shows that the commutativity on Δ -filtered objects implies the commutativity. \square

Let ${}^v O_\mu^{\text{proj}}$ and ${}^v \tilde{O}_\mu^{\text{proj}}$ be the full subcategories of projective modules in ${}^v O_\mu$ and ${}^v \tilde{O}_\mu$ respectively. The fully faithfulness of the functor \mathbb{V}_μ on projective modules implies the fully faithfulness of the functor $\tilde{\mathbb{V}}_\mu$ on projective modules. These functors identify ${}^v O_\mu^{\text{proj}}$ and ${}^v \tilde{O}_\mu^{\text{proj}}$ with some full subcategories in $\text{mod}({}^v Z_\mu)$ and $\text{grmod}({}^v Z_\mu)$ that we denote $\text{mod}({}^v Z_\mu)^{\text{proj}}$ and $\text{grmod}({}^v Z_\mu)^{\text{proj}}$ respectively. Since the functor F_k takes projective modules to projective modules, the commutativity of the diagram in Corollary 4.28 implies that the functor Res takes the category $\text{mod}({}^v Z_\mu)^{\text{proj}}$ to $\text{mod}({}^v Z_{\mu'})^{\text{proj}}$. This implies that its graded lift $\widetilde{\text{Res}}$ takes $\text{grmod}({}^v Z_\mu)^{\text{proj}}$ to $\text{grmod}({}^v Z_{\mu'})^{\text{proj}}$. Similar statements hold for Ind and $\widetilde{\text{Ind}}$.

Lemma 4.29. (a) *The functors E_k and F_k admit graded lifts $\tilde{E}_k: {}^v \tilde{O}_{\mu'} \rightarrow {}^v \tilde{O}_\mu$ and $\tilde{F}_k: {}^v \tilde{O}_\mu \rightarrow {}^v \tilde{O}_{\mu'}$. They can be chosen in such a way that the condition below holds.*

(b) *The following pairs of functors are adjoint*

$$(\tilde{F}_k, \tilde{E}_k \langle -\mu_{k+1} \rangle), \quad (\tilde{E}_k, \tilde{F}_k \langle \mu_{k+1} \rangle).$$

Proof. Let us prove (a). We give the prove only for the functor F_k . The proof for E_k is similar. The proof below is similar to the proof of [21, Lem. 5.10].

As explained above, the functor $\widetilde{\text{Res}}$ restricts to a functor $\text{grmod}({}^v Z_\mu)^{\text{proj}} \rightarrow \text{grmod}({}^v Z_{\mu'})^{\text{proj}}$. Together with the equivalences of categories ${}^v \tilde{O}_\mu^{\text{proj}} \simeq \text{grmod}({}^v Z_\mu)^{\text{proj}}$ and ${}^v \tilde{O}_{\mu'}^{\text{proj}} \simeq \text{grmod}({}^v Z_{\mu'})^{\text{proj}}$ obtained by restricting $\tilde{\mathbb{V}}_\mu$ and $\tilde{\mathbb{V}}_{\mu'}$ this yields a functor $\tilde{F}_k: {}^v \tilde{O}_\mu^{\text{proj}} \rightarrow {}^v \tilde{O}_{\mu'}^{\text{proj}}$. Next, we obtain a functor of homotopy categories $\tilde{F}_k: K^b({}^v \tilde{O}_\mu^{\text{proj}}) \rightarrow K^b({}^v \tilde{O}_{\mu'}^{\text{proj}})$. Since the categories ${}^v \tilde{O}_\mu$ and ${}^v \tilde{O}_{\mu'}$ have finite global dimensions, we have equivalences of categories $K^b({}^v \tilde{O}_\mu^{\text{proj}}) \simeq D^b({}^v \tilde{O}_\mu)$

and $K^b({}^v\tilde{O}_{\mu'}^{\text{proj}}) \simeq D^b({}^v\tilde{O}_{\mu'})$. Thus we get a functor of triangulated categories $\tilde{F}_k: D^b({}^v\tilde{O}_{\mu}) \rightarrow D^b({}^v\tilde{O}_{\mu'})$. If we repeat the same construction for non-graded categories, we obtain a functor $F_k: D^b({}^vO_{\mu}) \rightarrow D^b({}^vO_{\mu'})$ that is the same as the functor between the bounded derived categories induced by the exact functor $F_k: {}^vO_{\mu} \rightarrow {}^vO_{\mu'}$, see Corollary 4.28. This implies that the following diagram is commutative

$$\begin{array}{ccc} D^b({}^v\tilde{O}_{\mu}) & \xrightarrow{\tilde{F}_k} & D^b({}^v\tilde{O}_{\mu'}) \\ \text{forget} \downarrow & & \text{forget} \downarrow \\ D^b({}^vO_{\mu}) & \xrightarrow{F_k} & D^b({}^vO_{\mu'}) \end{array}$$

Since the bottom functor takes ${}^vO_{\mu}$ to ${}^vO_{\mu'}$, the top functor takes ${}^v\tilde{O}_{\mu}$ to ${}^v\tilde{O}_{\mu'}$. This completes the proof of (a).

Now we prove (b). The functors \tilde{E}_k and \tilde{F}_k are constructed as unique functors such that we have the following commutative diagrams

$$\begin{array}{ccccc} {}^vO_{\mu} & \xrightarrow{\tilde{F}_k} & {}^vO_{\mu'} & {}^vO_{\mu} & \xleftarrow{\tilde{E}_k} & {}^vO_{\mu'} \\ \tilde{\nabla}_{\mu} \downarrow & & \tilde{\nabla}_{\mu'} \downarrow & \tilde{\nabla}_{\mu} \downarrow & & \tilde{\nabla}_{\mu'} \downarrow \\ \text{mod}({}^vZ_{\mu'}) & \xrightarrow{\tilde{\text{Res}}} & \text{mod}({}^vZ_{\mu'}) & \text{mod}({}^vZ_{\mu}) & \xleftarrow{\tilde{\text{Ind}}} & \text{mod}({}^vZ_{\mu'}). \end{array} \quad (45)$$

By Corollary 4.25 (b) and Proposition 4.21 (b), the restrictions of the pairs $(\tilde{F}_k, \tilde{E}_k \langle -\mu_{k+1} \rangle)$ and $(\tilde{E}_k, \tilde{F}_k \langle \mu_{k+1} \rangle)$ to the subcategories of projective objects are biadjoint. We can conclude using the lemma below. \square

Lemma 4.30. *Let $\mathcal{C}_1, \mathcal{C}_2$ be abelian categories of finite global dimension and let $\mathcal{C}'_1, \mathcal{C}'_2$ be the full subcategories of projective objects. Assume that $E: \mathcal{C}_1 \rightarrow \mathcal{C}_2$, $F: \mathcal{C}_2 \rightarrow \mathcal{C}_1$ are exact functors. Assume that E and F send projective objects to projective objects and denote $E': \mathcal{C}'_1 \rightarrow \mathcal{C}'_2$, $F': \mathcal{C}'_2 \rightarrow \mathcal{C}'_1$ the restrictions of E and F . Assume that the pair (E', F') is adjoint. Then the pair (E, F) is adjoint.*

Proof. Let

$$\varepsilon': E'F' \rightarrow \text{Id}, \quad \eta': \text{Id} \rightarrow F'E'$$

be the counit and the unit of the adjoint pair (E', F') .

We can extend the functors E' and F' to functors $E': K^b(\mathcal{C}'_1) \rightarrow K^b(\mathcal{C}'_2)$ and $F': K^b(\mathcal{C}'_2) \rightarrow K^b(\mathcal{C}'_1)$ of the homotopy categories of bounded complexes. The counit ε' and the unit η' extend to natural transformations of functors of homotopy categories. These extended natural transformations still satisfy the properties of the counit and the unit of an adjunction. Thus the extended pair (E', F') is adjoint.

Since the categories \mathcal{C}_1 and \mathcal{C}_2 have finite global dimensions, we have equivalences of categories

$$K^b(\mathcal{C}'_1) \simeq D^b(\mathcal{C}_1), \quad K^b(\mathcal{C}'_2) \simeq D^b(\mathcal{C}_2). \quad (46)$$

By construction, the functors

$$E: D^b(\mathcal{C}_1) \rightarrow D^b(\mathcal{C}_2), \quad F: D^b(\mathcal{C}_2) \rightarrow D^b(\mathcal{C}_1) \quad (47)$$

obtained from functors E' and F' via the equivalences (46) coincide with the functors induced from $E: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ and $F: \mathcal{C}_2 \rightarrow \mathcal{C}_1$. The pair of functors (E, F) in (47) is adjoint with a counit ε and a unit η , obtained from ε' and η' . These counit and unit restrict to natural transformations of functors of abelian categories $E: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ and $F: \mathcal{C}_2 \rightarrow \mathcal{C}_1$. This proves the statement. \square

We need the following lemma later.

Lemma 4.31. *We have the following isomorphism of functors*

$$\widetilde{F}_k \widetilde{E}_k \simeq \text{Id}^{\oplus [\mu_{k+1}+1]_q} := \bigoplus_{r=0}^{\mu_{k+1}} \text{Id} \langle 2r - \mu_{k+1} \rangle,$$

where Id is the identity endofunctor of the category ${}^v\widetilde{\mathcal{O}}_{\mu'}$.

Proof. By Corollary 4.25 (c) we have $\widetilde{\text{Res}} \circ \widetilde{\text{Ind}} \simeq \text{Id}^{\oplus [\mu_{k+1}+1]_q}$. Then the diagrams (45) and Proposition 4.21 (b) yield an isomorphism $\widetilde{F}_k \widetilde{E}_k \simeq \text{Id}^{\oplus [\mu_{k+1}+1]_q}$ on projective modules in ${}^v\widetilde{\mathcal{O}}_{\mu'}$. This isomorphism can be extended to the category ${}^v\widetilde{\mathcal{O}}_{\mu'}$ in the same way as in the proof of Lemma 4.30. \square

4.9 The case $W_{\mu'} \subset W_{\mu}$

In the sections above we assumed $W_{\mu} \subset W_{\mu'}$ (or equivalently $\mu_k = 1$). In this section we announce similar results in the case $W_{\mu'} \subset W_{\mu}$ (or equivalently $\mu_{k+1} = 0$). All the proofs are the same as in the previous case but the roles of E_k and F_k should be exchanged.

Here we always assume that v is in $J_{\mu,+}$ (thus also in $J_{\mu',+}$). In contrast with the situation above, we have ${}^vZ_{\mu'} \subset {}^vZ_{\mu}$. Consider the induction and the restriction functors $\text{Ind}: \text{mod}({}^vZ_{\mu'}) \rightarrow \text{mod}({}^vZ_{\mu})$ and $\text{Res}: \text{mod}({}^vZ_{\mu}) \rightarrow \text{mod}({}^vZ_{\mu'})$.

Similarly to Corollary 4.28 we can prove the following statement.

Lemma 4.32. *The following diagrams of functors are commutative*

$$\begin{array}{ccc} {}^vO_{\mu} & \xrightarrow{F_k} & {}^vO_{\mu'} \\ \mathbb{V}_{\mu} \downarrow & & \mathbb{V}_{\mu'} \downarrow \\ \text{mod}({}^vZ_{\mu}) & \xrightarrow{\text{Ind}} & \text{mod}({}^vZ_{\mu'}), \\ {}^vO_{\mu} & \xleftarrow{E_k} & {}^vO_{\mu'} \\ \mathbb{V}_{\mu} \downarrow & & \mathbb{V}_{\mu'} \downarrow \\ \text{mod}({}^vZ_{\mu}) & \xleftarrow{\text{Res}} & \text{mod}({}^vZ_{\mu'}). \end{array}$$

\square

Next, similarly to Lemmas 4.29, 4.31 we can deduce the following result.

Lemma 4.33. (a) The functors E_k and F_k admit graded lifts $\tilde{E}_k: {}^v\tilde{O}_{\mu'} \rightarrow {}^v\tilde{O}_{\mu}$. They can be chosen in such a way that the conditions below hold.
(b) The following pairs of functors are adjoint

$$(\tilde{F}_k, \tilde{E}_k\langle\mu_k - 1\rangle), \quad (\tilde{E}_k, \tilde{F}_k\langle-\mu_k + 1\rangle).$$

(c) We have the following isomorphism of functors

$$\tilde{E}_k\tilde{F}_k \simeq \text{Id}^{\oplus[\mu_k]_q} := \bigoplus_{r=0}^{\mu_k-1} \text{Id}\langle 2r - \mu_k + 1\rangle,$$

where Id is the identity endofunctor of the category ${}^v\tilde{O}_{\mu}$.

5 Koszul duality

5.1 Bimodules over a semisimple basic algebra

Let B be a \mathbb{C} -algebra isomorphic to a finite direct sum of copies of \mathbb{C} . We have $B = \bigoplus_{\lambda \in \Lambda} \mathbb{C}e_{\lambda}$ for some idempotents e_{λ} .

Definition 5.1. Let $\text{bmod}(B)$ be the category of finite dimensional (B, B) -bimodules.

A bimodule $M \in \text{bmod}(B)$ can be viewed just as a collection of finite dimensional \mathbb{C} -vector spaces $e_{\lambda}Me_{\mu}$ for $\lambda, \mu \in \Lambda$. To each bimodule $M \in \text{bmod}(B)$ we can associate a bimodule $M^{\bullet} \in \text{bmod}(B)$ as follows $M^{\bullet} = \text{Hom}_{\text{bmod}(B)}(M, B \otimes_{\mathbb{C}} B)$. The bimodule structure on M^{\bullet} is defined in the following way. For $f \in M^{\bullet}$, $m \in M$, $b_1, b_2 \in B$ we have $b_1fb_2(m) = f(b_2mb_1)$.

Lemma 5.2. Assume that $M, N \in \text{bmod}(B)$, $X, Y \in \text{mod}(B)$, $Z \in \text{mod}(B)^{\text{op}}$. Then we have the following isomorphisms:

- (a) $\text{Hom}_{\text{bmod}(B)}(M, N) \simeq \bigoplus_{\lambda, \mu \in \Lambda} \text{Hom}_{\mathbb{C}}(e_{\lambda}Me_{\mu}, e_{\lambda}Me_{\mu})$,
- (b) $\text{Hom}_B(X, Y) \simeq \bigoplus_{\lambda \in \Lambda} \text{Hom}_{\mathbb{C}}(e_{\lambda}X, e_{\lambda}Y)$,
- (c) $X \otimes_B Z = \bigoplus_{\lambda \in \Lambda} X e_{\lambda} \otimes_{\mathbb{C}} e_{\lambda}Z$,
- (d) $e_{\lambda}M^{\bullet}e_{\mu} \simeq (e_{\mu}Me_{\lambda})^{\bullet}$, where \bullet^{\bullet} is the usual duality for \mathbb{C} -vector spaces,
- (e) $\text{Hom}_B(M^{\bullet} \otimes_B X, Y) \simeq \text{Hom}_B(X, M \otimes_B Y)$,
- (f) $(M \otimes_B N)^{\bullet} \simeq N^{\bullet} \otimes_B M^{\bullet}$.

Proof. Parts (a), (b), (c) are obvious. Part (d) follows from (a). Part (e) follows from (b), (c), (d). Part (f) follows from (c), (d). \square

5.2 Quadratic dualities

Let $A = \bigoplus_{n \in \mathbb{N}} A_n$ be a finite dimensional \mathbb{N} -graded algebra over \mathbb{C} . Assume that A_0 is semisimple and basic. Let $T_{A_0}(A_1) = \bigoplus_{n \in \mathbb{N}} A_1^{\otimes n}$ be the tensor algebra of A_1 over A_0 , here $A_1^{\otimes n}$ means $A_1 \otimes_{A_0} A_1 \otimes_{A_0} \cdots \otimes_{A_0} A_1$ with n components A_1 . The algebra A is said to be *quadratic* if it is generated by elements of degree 0 and 1 with relations in degree 2, i.e., the kernel of the obvious map $T_{A_0}(A_1) \rightarrow A$ is generated by elements in $A_1 \otimes_{A_0} A_1$.

Definition 5.3. Consider the (A_0, A_0) -bimodule morphism $\phi: A_1 \otimes_{A_0} A_1 \rightarrow A_2$ given by the product in A . Let $\phi^*: A_2^* \rightarrow A_1^* \otimes_{A_0} A_1^*$ be the dual morphism to ϕ , see Lemma 5.2, here \bullet^* is as in Section 5.1 with respect to $B = A_0$. The *quadratic dual algebra* to A is the quadratic algebra $A^! = T_{A_0}(A_1^*)/(\text{Im } \phi^*)$.

Remark 5.4. In the previous definition we do not assume that the algebra A is quadratic itself. However, if it is true, we have a graded \mathbb{C} -algebra isomorphism $(A^!)^! \simeq A$.

Let \mathcal{C} be an abelian category such that its objects are graded modules. Denote by $\text{Com}^\downarrow(\mathcal{C})$ the category of complexes X^\bullet in \mathcal{C} such that the j th graded component of X^i is zero when $i \gg 0$ or $i + j \ll 0$. Similarly, let $\text{Com}^\uparrow(\mathcal{C})$ the category of complexes X^\bullet in \mathcal{C} such that the j th graded component of X^i is zero when $i \ll 0$ or $i + j \gg 0$. Denote by $D^\downarrow(\mathcal{C})$ and $D^\uparrow(\mathcal{C})$ the corresponding derived categories of such complexes. We will use the following abbreviations

$$D^\downarrow(A) = D^\downarrow(\text{grmod}(A)), \quad D^\uparrow(A) = D^\uparrow(\text{grmod}(A)), \quad D^b(A) = D^b(\text{grmod}(A)).$$

In the situation above we have the following functors $\mathcal{K}: D^\downarrow(A) \rightarrow D^\uparrow(A^!)$ and $\mathcal{K}': D^\uparrow(A^!) \rightarrow D^\downarrow(A)$ called *quadratic duality functors*. See [16, Sec. 5] for more details.

5.3 Koszul algebras

Let $A = \bigoplus_{n \in \mathbb{N}} A_n$ be a finite dimensional \mathbb{N} -graded \mathbb{C} -algebra such that A_0 is semisimple. We identify A_0 with the left graded A -module $A_0 \simeq A/\bigoplus_{n>0} A_n$.

Definition 5.5. The graded algebra A is *Koszul* if the left graded A -module A_0 admits a projective resolution $\cdots \rightarrow P^2 \rightarrow P^1 \rightarrow P^0 \rightarrow A_0$ such that P^r is generated by its degree r component.

If A is Koszul, we consider the graded \mathbb{C} -algebra $A^! = \text{Ext}_A^*(A_0, A_0)^{\text{op}}$ and we call it the *Koszul dual algebra* to A . The following is well-known, see [3].

Proposition 5.6. *Let A be a Koszul \mathbb{C} -algebra. Assume that A and $A^!$ are finite dimensional. Then, the following holds.*

- (a) *The algebra A is quadratic. The Koszul dual algebra $A^!$ coincides with the quadratic dual algebra.*
- (b) *The algebra $A^!$ is also Koszul and there is a graded algebra isomorphism $(A^!)^! \simeq A$.*

(c) *There is an equivalence of categories*

$$\mathcal{K}: D^b(A) \rightarrow D^b(A^\dagger), \quad M \mapsto \mathrm{RHom}_A(A_0, M).$$

□

If A is Koszul, then the functors \mathcal{K} and \mathcal{K}' from the previous section are mutually inverse. Moreover, the equivalence \mathcal{K} of bounded derived categories in Proposition 5.6 (c) is the restriction of the functor \mathcal{K} from the previous section.

Definition 5.7. Let A and B be Koszul algebras. We say that the functor $\Phi: D^b(A) \rightarrow D^b(B)$ is Koszul dual to the functor $\Psi: D^b(A^\dagger) \rightarrow D^b(B^\dagger)$ if the following diagram of functor is commutative

$$\begin{array}{ccc} D^b(A) & \xrightarrow{\Psi} & D^b(B) \\ \kappa \downarrow & & \downarrow \kappa \\ D^b(A^\dagger) & \xrightarrow{\Phi} & D^b(B^\dagger). \end{array}$$

5.4 Categories of linear complexes

In this section we recall some results from [16] about linear complexes. Let A be as in Section 5.2.

Definition 5.8. Let $\mathcal{LC}(A)$ be the category of complexes $\cdots \rightarrow \mathcal{X}^{k-1} \rightarrow \mathcal{X}^k \rightarrow \mathcal{X}^{k+1} \rightarrow \cdots$ of projective modules in $\mathrm{grmod}(A)$ such that for each $k \in \mathbb{Z}$ each indecomposable direct factor P of \mathcal{X}^k is a direct factor of $A\langle k \rangle$.

Proposition 5.9. *There is an equivalence of categories $\epsilon_A: \mathcal{LC}(A) \simeq \mathrm{grmod}(A^\dagger)$.*

□

Let us describe the construction of ϵ_A^{-1} . Let $M = \oplus_{n \in \mathbb{Z}} M_n$ be in $\mathrm{grmod}(A^\dagger)$. The graded A^\dagger -module structure yields morphisms of A_0 -modules $f'_n: A_1^\dagger \otimes M_n \rightarrow M_{n+1}$ for each $n \in \mathbb{Z}$. We have

$$\begin{aligned} \mathrm{Hom}_{A_0}(A_1^\dagger \otimes_{A_0} M_n, M_{n+1}) &= \mathrm{Hom}_{A_0}(M_n, (A_1^\dagger)^\star \otimes_{A_0} M_{n+1}) \\ &= \mathrm{Hom}_{A_0}(M_n, A_1 \otimes_{A_0} M_{n+1}). \end{aligned}$$

Let $f_n: \mathrm{Hom}_{A_0}(M_n, A_1 \otimes_{A_0} M_{n+1})$ be the image of f'_n by the chain of isomorphisms above.

We have $\epsilon_A^{-1}(M) = \cdots \xrightarrow{\partial_{k-2}} \mathcal{X}^{k-1} \xrightarrow{\partial_{k-1}} \mathcal{X}^k \xrightarrow{\partial_k} \mathcal{X}^{k+1} \xrightarrow{\partial_{k+1}} \cdots$ with $\mathcal{X}^k = A\langle k \rangle \otimes_{A_0} M_k$ and

$$\partial_k: A\langle k \rangle \otimes_{A_0} M_k \rightarrow A\langle k+1 \rangle \otimes_{A_0} M_{k+1}, \quad a \otimes m \mapsto (a \otimes \mathrm{Id})(f_k(m)).$$

The quadratic duality functor discussed in the previous section can be characterized as follows, see [16, Prop. 21].

Lemma 5.10. *Up to isomorphism of functors, the following diagram is commutative:*

$$\begin{array}{ccc}
 & D^\uparrow(\mathcal{LC}(A)) & \\
 \text{Tot} \swarrow & & \nwarrow \epsilon_A^{-1} \\
 D^\downarrow(A) & \xleftarrow{\mathcal{K}'} & D^\uparrow(A^!)
 \end{array} ,$$

where Tot is the functor taking the total complex. □

5.5 The main lemma about Koszul dual functors

Let $\{e_\lambda; \lambda \in \Lambda\}$ be the set of indecomposable idempotents of A_0 , i.e., we have $A_0 = \bigoplus_{\lambda \in \Lambda} \mathbb{C}e_\lambda$. Denote by $e_\lambda^!$ the corresponding idempotent of $A_0^!$ via the identification $A_0 \simeq A_0^!$. For each subset $\Lambda' \subset \Lambda$ set $e_{\Lambda'} = \sum_{\lambda \in \Lambda'} e_\lambda$. Consider the graded algebras

$$A_{\Lambda'} = e_{\Lambda'} A e_{\Lambda'}, \quad {}_{\Lambda'} A = A / (e_{\Lambda \setminus \Lambda'}).$$

Similarly, we can define $A_{\Lambda'}^!$ and ${}_{\Lambda'} A^!$.

We have a functor $F: \text{grmod}(A_{\Lambda'}) \rightarrow \text{grmod}(A)$, $M \mapsto A e_{\Lambda'} \otimes_{A_{\Lambda'}} M$. Note also that the category $\text{grmod}({}_{\Lambda'} A^!)$ can be viewed as a subcategory of $\text{grmod}(A^!)$ containing modules that are killed by $e_{\Lambda \setminus \Lambda'}$. Let $\iota: \text{grmod}({}_{\Lambda'} A^!) \rightarrow \text{grmod}(A^!)$ be the inclusion. The following proposition is proved in [16, Thm. 28].

Proposition 5.11. (a) *The quadratic dual algebra to $A_{\Lambda'}$ is isomorphic to ${}_{\Lambda'} A^!$.*
 (b) *The following diagram commutes up to isomorphism of functors.*

$$\begin{array}{ccc}
 D^\downarrow(A) & \xleftarrow{\mathcal{K}'} & D^\uparrow(A^!) \\
 F \uparrow & & \uparrow \iota \\
 D^\downarrow(A_{\Lambda'}) & \xleftarrow{\mathcal{K}'} & D^\uparrow({}_{\Lambda'} A^!)
 \end{array}$$

Idea of proof of (b). By Lemma 5.10 it is enough to proof the commutativity of the following diagram.

$$\begin{array}{ccc}
 \mathcal{LC}(A) & \xleftarrow{\epsilon_A^{-1}} & \text{grmod}(A^!) \\
 F \uparrow & & \uparrow \iota \\
 \mathcal{LC}(A_{\Lambda'}) & \xleftarrow{\epsilon_{A_{\Lambda'}}^{-1}} & \text{grmod}({}_{\Lambda'} A^!)
 \end{array}$$

□

We can generalize this result as follows.

Lemma 5.12. *Let A' be a finite dimensional \mathbb{N} -graded \mathbb{C} -algebra. Assume that for some subset $\Lambda' \subset \Lambda$ there is a graded (unitary) homomorphism $\psi: A' \rightarrow A_{\Lambda'}$ such that*

- (a) ψ is an isomorphism in degrees 0 and 1,
- (b) ψ induces an isomorphism between the kernel of $A'_1 \otimes_{A'_0} A'_1 \rightarrow A'_2$ and the kernel of $(A_{\Lambda'})_1 \otimes_{(A_{\Lambda'})_0} (A_{\Lambda'})_1 \rightarrow (A_{\Lambda'})_2$.

Then the quadratic dual of A' is isomorphic to ${}_{\Lambda'} A$.

Consider the graded (A, A') -bimodule $Ae_{\Lambda'}$, where the right A' -module structure is obtained from the right $A_{\Lambda'}$ -module structure using ψ . Consider the functor $T: \text{grmod}(A') \rightarrow \text{grmod}(A)$, $M \mapsto Ae_{\Lambda'} \otimes_{A'} M$. Then the following diagram commutes up to an isomorphism of functors.

$$\begin{array}{ccc} D^\downarrow(A) & \xleftarrow{\kappa'} & D^\uparrow(A^\dagger) \\ T \uparrow & & \iota \uparrow \\ D^\downarrow(A') & \xleftarrow{\kappa'} & D^\uparrow({}_{\Lambda'} A^\dagger) \end{array}$$

Proof. By definition, the quadratic dual of A' depends only on the algebra A'_0 , the (A'_0, A'_0) -bimodule A'_1 and the kernel of $A'_1 \otimes_{A'_0} A'_1 \rightarrow A'_2$. Thus the quadratic dual algebras of A' and $A_{\Lambda'}$ are isomorphic. Finally, Proposition 5.11 (a) implies that the quadratic dual of A' is isomorphic to ${}_{\Lambda'} A$.

Now, by Lemma 5.10 is enough to prove the commutativity of the following diagram up to an isomorphism of functors.

$$\begin{array}{ccc} \mathcal{LC}(A) & \xleftarrow{\epsilon_A^{-1}} & \text{grmod}(A^\dagger) \\ T \uparrow & & \iota \uparrow \\ \mathcal{LC}(A') & \xleftarrow{\epsilon_{A'}^{-1}} & \text{grmod}({}_{\Lambda'} A^\dagger) \end{array}$$

By analogy with the definition of the functor T , consider the functor $\Phi: \text{grmod}(A') \rightarrow \text{grmod}(A_{\Lambda'})$, $M \mapsto A_{\Lambda'} \otimes_{A'} M$. For each $\lambda \in \Lambda'$ let e'_λ be the idempotent in A'_0 such that $\psi(e'_\lambda) = e_\lambda$. We have $\Phi(A'e'_\lambda) = A_{\Lambda'}e_\lambda$ for each $\lambda \in \Lambda'$. In particular Φ induces a bijection between the indecomposable direct factors of A' and $A_{\Lambda'}$. Thus Φ induces a functor $\Phi: \mathcal{LC}(A') \rightarrow \mathcal{LC}(A_{\Lambda'})$. Note that by definition the boundary maps in the complexes of the category $\mathcal{LC}(\bullet)$ are of degree 1. Thus, by (a) and (b) the functor Φ induces an equivalence of categories $\Phi: \mathcal{LC}(A') \rightarrow \mathcal{LC}(A_{\Lambda'})$.

Consider the following diagram, where the functor F is as before Proposition 5.11.

$$\begin{array}{ccccc} \mathcal{LC}(A) & \xleftarrow{\text{Id}} & \mathcal{LC}(A) & \xleftarrow{\epsilon_A^{-1}} & \text{grmod}(A^\dagger) \\ T \uparrow & & F \uparrow & & \iota \uparrow \\ \mathcal{LC}(A') & \xleftarrow{\Phi^{-1}} & \mathcal{LC}(A_{\Lambda'}) & \xleftarrow{\epsilon_{A_{\Lambda'}}^{-1}} & \text{grmod}({}_{\Lambda'} A^\dagger) \end{array}$$

The right square commutes by the proof of Proposition 5.11 and the commutativity of the left square is obvious. To conclude we need only to check that $\epsilon_{A'}^{-1} = \Phi^{-1} \circ \epsilon_{A_{\Lambda'}}^{-1}$.

Let us check that $\Phi \circ \epsilon_{A'}^{-1} = \epsilon_{A_{\Lambda'}}^{-1}$. This is clear on objects because

$$\begin{array}{ccccccc} \epsilon_{A_{\Lambda'}}^{-1}(M) & = \cdots & \xrightarrow{\partial'_{k-1}} & A_{\Lambda'} \langle k \rangle \otimes_{(A_{\Lambda'})_0} M_k & \xrightarrow{\partial'_k} & A_{\Lambda'} \langle k+1 \rangle \otimes_{(A_{\Lambda'})_0} M_{k+1} & \xrightarrow{\partial'_{k+1}} \cdots, \\ \epsilon_{A'}^{-1}(M) & = \cdots & \xrightarrow{\partial''_{k-1}} & A' \langle k \rangle \otimes_{A'_0} M_k & \xrightarrow{\partial''_k} & A' \langle k+1 \rangle \otimes_{A'_0} M_{k+1} & \xrightarrow{\partial''_{k+1}} \cdots. \end{array}$$

The boundary maps are defined as follows

$$\begin{array}{ll} \partial'_k: A_{\Lambda'} \langle k \rangle \otimes_{(A_{\Lambda'})_0} M_k \rightarrow A_{\Lambda'} \langle k+1 \rangle \otimes_{(A_{\Lambda'})_0} M_{k+1}, & a \otimes m \mapsto (a \otimes \text{Id})(f_n^1(m)), \\ \partial''_k: A' \langle k \rangle \otimes_{A'_0} M_k \rightarrow A' \langle k+1 \rangle \otimes_{A'_0} M_{k+1}, & a \otimes m \mapsto (a \otimes \text{Id})(f_n^2(m)), \end{array}$$

where $f_n^1: M_n \rightarrow (A_{\Lambda'})_1 \otimes_{(A_{\Lambda'})_0} M_{n+1}$ and $f_n^2: M_n \rightarrow A'_1 \otimes_{A'_0} M_{n+1}$ are defined in the same way as f_n in the definition of ϵ^{-1} . Thus it is also clear that Φ commutes with the boundary maps. \square

Remark 5.13. Condition (b) is necessary only to deduce that $(A')^! \simeq (A_{\Lambda'})^!$. Without this condition we know only that the algebra $(A')^!$ is isomorphic to a quotient of $(A_{\Lambda'})^!$. Thus condition (b) can be replaced by the requirement $\dim(A')^! = \dim_{\Lambda'} A^!$.

We can reformulate Lemma 5.12 in the following way.

Corollary 5.14. *Let A' be an \mathbb{N} -graded finite dimensional \mathbb{C} -algebra with basic A'_0 such that the indecomposable idempotents of A'_0 are parameterized by a subset Λ' of Λ , i.e., we have $A'_0 = \bigoplus_{\lambda \in \Lambda'} \mathbb{C}e'_\lambda$. Assume that $\dim(A')^! = \dim_{\Lambda'} A^!$. Assume also that there is an exact functor $T: \text{grmod}(A') \rightarrow \text{grmod}(A)$ such that*

- (a) $T(A'e'_\lambda) = Ae_\lambda \ \forall \lambda \in \Lambda'$,
- (b) *the functor T yields an isomorphism $\text{Hom}_{A'}(A'e'_\lambda \langle 1 \rangle, A'e'_\mu) \simeq \text{Hom}_A(Ae_\lambda \langle 1 \rangle, Ae_\mu)$.*

Then the quadratic dual for A' is ${}_{\Lambda'} A^!$ and the following diagram commutes up to isomorphism of functors.

$$\begin{array}{ccc} D^\downarrow(A) & \xleftarrow{\kappa'} & D^\uparrow(A^!) \\ T \uparrow & & \iota \uparrow \\ D^\downarrow(A') & \xleftarrow{\kappa'} & D^\uparrow({}_{\Lambda'} A^!) \end{array}$$

Proof. Condition (a) implies that the functor T yields a homomorphism of graded algebras $\psi: A' \rightarrow A_{\Lambda'}$. Moreover, condition (b) implies that ψ satisfies condition (a) of Lemma 5.12. Finally, the assumption $\dim(A')^! = \dim_{\Lambda'} A^!$ implies that ψ satisfies condition (b) of Lemma 5.12, see Remark 5.13. The functor T here can be identified with the functor $T = Ae_{\Lambda'} \otimes_{A'} \bullet$ in the statement of Lemma 5.12, see [22, Lem. 3.4]. Thus the statement follows from Lemma 5.12. \square

5.6 Zuckerman functors

Fix $v \in \widehat{W}$. Let ν_1 and ν_2 be two different parabolic types such that $W_{\nu_1} \subset W_{\nu_2}$. By definition of the parabolic category \mathcal{O} , there is an inclusion of categories ${}^v\mathcal{O}_{\mu}^{\nu_2} \subset {}^v\mathcal{O}_{\mu}^{\nu_1}$. We denote by inc the inclusion functor. We may write $\text{inc} = \text{inc}_{\nu_2}^{\nu_1}$ to specify the parameters. The functor inc admits a left adjoint functor tr . For $M \in {}^v\mathcal{O}_{\mu}^{\nu_1}$, the object $\text{tr}(M)$ is the maximal quotient of M that is in ${}^v\mathcal{O}_{\mu}^{\nu_2}$, see Lemma 4.13 (a). We call the functor tr the *parabolic truncation* functor. We may write $\text{tr}_{\nu_1}^{\nu_2}$ to specify the parameters.

Now, we assume that ν_1 and ν_2 are two arbitrary parabolic types. Then there is a parabolic type ν_3 such that we have $W_{\nu_3} = W_{\nu_1} \cap W_{\nu_2}$. The *Zuckerman functor* $\text{Zuc}_{\nu_1}^{\nu_2}$ (or simply Zuc) is the composition $\text{Zuc}_{\nu_1}^{\nu_2} = \text{tr}_{\nu_3}^{\nu_2} \circ \text{inc}_{\nu_1}^{\nu_3}$.

The parabolic inclusion functor is exact. The parabolic truncation functor is only right exact. This implies that the Zuckerman functor is right exact.

Now, we are going to grade Zuckerman functors. Let ${}^vA_{\mu}^{\nu}$ be the indomorphism algebra of the minimal projective generator of ${}^v\mathcal{O}_{\mu}^{\nu}$ (or simply ${}^vA_{\mu}$ in the non-parabolic case). We have ${}^v\mathcal{O}_{\mu}^{\nu} \simeq \text{mod}({}^vA_{\mu}^{\nu})$. The Koszul grading on ${}^vA_{\mu}^{\nu}$ is constructed in [21]. The graded version ${}^v\widetilde{\mathcal{O}}_{\mu}^{\nu}$ of ${}^v\mathcal{O}_{\mu}^{\nu}$ is the category $\text{grmod}({}^vA_{\mu}^{\nu})$. Moreover, the algebra ${}^vA_{\mu}^{\nu}$ is the quotient of ${}^vA_{\mu}$ by a homogeneous ideal I_{ν} . By construction, the grading on ${}^vA_{\mu}^{\nu}$ is induced from the grading on ${}^vA_{\mu}$. Assume that ν_1 and ν_2 are such that $W_{\nu_1} \subset W_{\nu_2}$. Then we have $I_{\nu_1} \subset I_{\nu_2}$. This implies that the graded algebra ${}^vA_{\mu}^{\nu_2}$ is isomorphic to the quotient of the graded algebra ${}^vA_{\mu}^{\nu_1}$ by the homogeneous ideal I_{ν_2}/I_{ν_1} . This yields an inclusion of graded categories ${}^v\widetilde{\mathcal{O}}_{\mu}^{\nu_2} \subset {}^v\widetilde{\mathcal{O}}_{\mu}^{\nu_1}$. Let us denote by $\widetilde{\text{inc}}_{\nu_2}^{\nu_1}$ (or simply $\widetilde{\text{inc}}$) the inclusion functor. It is a graded lift of the functor inc . Similarly, its left adjoint functor $\widetilde{\text{tr}}_{\nu_1}^{\nu_2}$ is a graded lift of the functor tr , see Remark 4.14. Thus we get graded lifts $\widetilde{\text{Zuc}}_{\nu_1}^{\nu_2}$ of the Zuckerman functor $\text{Zuc}_{\nu_1}^{\nu_2}$ for arbitrary parabolic types ν_1 and ν_2 .

Similarly, we can define the parabolic inclusion functor, the parabolic truncation functor, the Zuckerman functor and their graded versions for the affine category \mathcal{O} at a positive level.

5.7 The Koszul dual functors in the category \mathcal{O}

As above, we assume $W_{\mu} \subset W_{\mu'}$. Set $J_{\mu}^{\nu} = \{w \in J_{\mu}; w(1_{\mu}) \in P^{\nu}\}$. Note that the inclusion $J_{\mu'} \subset J_{\mu}$ induces an inclusion $J_{\mu'}^{\nu} \subset J_{\mu}^{\nu}$. For $v \in \widehat{W}$ we set ${}^vJ_{\mu}^{\nu} = \{w \in J_{\mu}^{\nu}; w \leq v\}$.

As in Section 4 we assume that we have $W_{\mu} \subset W_{\mu'}$. Fix a parabolic type $\nu = (\nu_1, \dots, \nu_l) \in X_l[N]$.

Assume $v \in J_{\mu'}^{\nu} w_{\mu'}$. The functors

$$F_k: {}^v\mathcal{O}_{\mu} \rightarrow {}^v\mathcal{O}_{\mu'}, \quad E_k: {}^v\mathcal{O}_{\mu'} \rightarrow {}^v\mathcal{O}_{\mu}$$

restrict to functors of parabolic categories

$$F_k: {}^v\mathcal{O}_{\mu}^{\nu} \rightarrow {}^v\mathcal{O}_{\mu'}^{\nu}, \quad E_k: {}^v\mathcal{O}_{\mu'}^{\nu} \rightarrow {}^v\mathcal{O}_{\mu}^{\nu}.$$

The restricted functors still satisfy the properties announced in Lemmas 4.29, 4.31.

Assume that $w \in {}^v J_\mu^\nu$. Let ${}^v P^{w(1_\mu)}$ be the projective cover of $L^{w(1_\mu)}$ in ${}^v \mathcal{O}_\mu^\nu$. (Note that we do not indicate the parabolic type ν in our notations for modules to simplify the notations.) We fix the grading on $L^{w(1_\mu)}$ such that it is concentrated in degree zero when we consider $L^{w(1_\mu)}$ as an ${}^v A_\mu^\nu$ -module (see Section 5.6 for the definition of ${}^v A_\mu^\nu$). A standard argument shows that the modules ${}^v P^{w(1_\mu)}$ and $\Delta^{w(1_\mu)}$ admit graded lifts. (The graded lift of ${}^v P^{w(1_\mu)}$ can be constructed as the projective cover of the graded lift of $L^{w(1_\mu)}$ in ${}^v \tilde{\mathcal{O}}_\mu^\nu$. The existence of graded lifts of projective modules implies the existence of graded lifts of Verma modules, see [15, Cor. 4].) We fix the graded lifts of ${}^v P^{w(1_\mu)}$ and $\Delta^{w(1_\mu)}$ such that the surjections ${}^v P^{w(1_\mu)} \rightarrow L^{w(1_\mu)}$ and $\Delta^{w(1_\mu)} \rightarrow L^{w(1_\mu)}$ are homogeneous of degree zero, see also Lemma 4.1.

The following lemma is stated in the parabolic category \mathcal{O} .

Lemma 5.15. (a) For each $w \in {}^v J_{\mu'}^\nu$, we have $E_k({}^v P^{w(1_{\mu'})}) = {}^v P^{w(1_\mu)}$.
(b) For each $w \in {}^v J_\mu^\nu$, we have

$$F_k(L^{w(1_\mu)}) = \begin{cases} L^{w(1_{\mu'})} & \text{if } w \in {}^v J_{\mu'}^\nu, \\ 0 & \text{else.} \end{cases}$$

Proof. First, we prove (a) in the non-parabolic situation (i.e., for $\nu = (1, 1, \dots, 1)$). The modules $E_k({}^v P^{w(1_{\mu'})})$ and ${}^v P^{w(1_\mu)}$ are both projective. Thus it is enough to show that their classes in the Grothendieck group are the same. To show this, we compare the multiplicities of Verma modules in the Δ -filtrations of $E_k({}^v P^{w(1_{\mu'})})$ and ${}^v P^{w(1_\mu)}$.

We need to show that for each $x \in {}^v J_{\mu'}$ we have

$$[E_k({}^v P^{w(1_{\mu'})}), \Delta^{x(1_{\mu'})}] = [{}^v P^{w(1_\mu)}, \Delta^{x(1_\mu)}].$$

By Lemma 4.18, for each $x \in {}^v J_\mu$, the multiplicity $[E_k({}^v P^{w(1_{\mu'})}), \Delta^{x(1_\mu)}]$ is equal to the multiplicity $[{}^v P^{w(1_{\mu'})}, \Delta^{x(1_{\mu'})}]$. So, we need to prove the equality

$$[{}^v P^{w(1_{\mu'})}, \Delta^{x(1_{\mu'})}] = [{}^v P^{w(1_\mu)}, \Delta^{x(1_\mu)}].$$

The last equality is obvious because both of these multiplicities are given by the same parabolic Kazhdan-Lusztig polynomial. See, for example, [13, App. A] for more details about multiplicities in the parabolic category \mathcal{O} for $\widehat{\mathfrak{gl}}_N$.

Now, we prove (b). Since the set of simple modules in the parabolic category \mathcal{O} is a subset of the set of simple modules of the non-parabolic case category \mathcal{O} , it is enough to prove (b) in the non-parabolic case.

For each $w \in {}^v J_\mu$ and $x \in {}^v J_{\mu'}$, we have

$$\begin{aligned} \text{Hom}({}^v P^{x(1_{\mu'})}, F_k(L^{w(1_\mu)})) &\simeq \text{Hom}(E_k({}^v P^{x(1_{\mu'})}), L^{w(1_\mu)}) \\ &\simeq \text{Hom}({}^v P^{x(1_\mu)}, L^{w(1_\mu)}). \end{aligned}$$

This implies that we have $\dim \operatorname{Hom}(^v P^{x(1_{\mu'})}, F_k(L^{w(1_{\mu})})) = \delta_{x,w}$. Since $\dim \operatorname{Hom}(^v P^{x(1_{\mu'})}, M)$ counts the multiplicity of the simple module $L^{x(1_{\mu'})}$ in the module M (this fact can be proved in the same way as [11, Thm. 3.9 (c)]), this proves (b).

Finally, we prove (a) in the parabolic situation. For each $w \in {}^v J_{\mu'}^{\nu}$ and each $x \in {}^v J_{\mu}^{\nu}$ we have

$$\begin{aligned} \operatorname{Hom}(E_k(^v P^{w(1_{\mu'})}), L^{x(1_{\mu})}) &\simeq \operatorname{Hom}(^v P^{w(1_{\mu'})}, F_k(L^{x(1_{\mu})})) \\ &\simeq \begin{cases} \operatorname{Hom}(^v P^{w(1_{\mu'})}, L^{x(1_{\mu'})}) & \text{if } x \in {}^v J_{\mu'}^{\nu} \\ 0 & \text{else,} \end{cases} \end{aligned}$$

where the second isomorphism follows from (b). This implies that we have $\dim \operatorname{Hom}(E_k(^v P^{w(1_{\mu'})}), L^{x(1_{\mu})}) = \delta_{w,x}$. Thus we have $E_k(^v P^{w(1_{\mu'})}) \simeq {}^v P^{w(1_{\mu})}$. \square

The definitions of the graded lifts \tilde{E}_k and \tilde{F}_k in Lemma 4.29 depend on the choice of the graded lift $\tilde{\mathbb{V}}_{\mu}$ of \mathbb{V}_{μ} . Note that we have the following isomorphism of ${}^v Z_{\mu}$ -modules $\mathbb{V}_{\mu}(^v P^{\mu}) \simeq {}^v Z_{\mu}$ for all $\mu \in X_I[N]$. By Lemma 4.1, for each choice of the graded lift $\tilde{\mathbb{V}}_{\mu}$, we have $\tilde{\mathbb{V}}_{\mu}(^v P^{\mu}) \simeq {}^v Z_{\mu}\langle r \rangle$ for some $r \in \mathbb{Z}$. From now on, we always assume that the graded lift $\tilde{\mathbb{V}}_{\mu}$ is chosen in such a way that we have an isomorphism of graded ${}^v Z_{\mu}$ -modules $\tilde{\mathbb{V}}_{\mu}(^v P^{\mu}) \simeq {}^v Z_{\mu}$ (without any shift r).

In the following statement we consider the non-parabolic situation.

Lemma 5.16. *For each $w \in {}^v J_{\mu'}$, the graded module $\tilde{E}_k(\Delta^{w(1_{\mu'})})$ has a graded Δ -filtration with constituents $\Delta^{wz(1_{\mu})}\langle \ell(z) \rangle$ for $z \in J_{\mu'}^{\mu}$.*

Proof. First, we prove that \tilde{E}_k takes the graded anti-dominant projective module to the graded anti-dominant projective module, i.e., that we have $\tilde{E}_k(^v P^{\mu'}) \simeq {}^v P^{\mu}$.

By Lemma 4.1, the graded lift of ${}^v P^{\mu}$ is unique up to graded shift. Thus, by Lemma 5.15, we have $\tilde{E}_k(^v P^{\mu'}) = {}^v P^{\mu}\langle r \rangle$ for some $r \in \mathbb{Z}$. We need to prove that $r = 0$.

Recall that the graded lift \tilde{E}_k of E_k is constructed in the proof of Lemma 4.29 in such a way that the following diagram is commutative

$$\begin{array}{ccc} {}^v O_{\mu} & \xleftarrow{\tilde{E}_k} & {}^v O_{\mu'} \\ \tilde{\mathbb{V}}_{\mu} \downarrow & & \tilde{\mathbb{V}}_{\mu'} \downarrow \\ \operatorname{mod}({}^v Z_{\mu}) & \xleftarrow{\widetilde{\operatorname{Ind}}} & \operatorname{mod}({}^v Z_{\mu'}). \end{array}$$

Moreover, by definition, we have the following isomorphisms of graded modules

$$\tilde{\mathbb{V}}_{\mu}(^v P^{w(1_{\mu})}) \simeq {}^v Z_{\mu}, \quad \tilde{\mathbb{V}}_{\mu'}(^v P^{w(1_{\mu'})}) \simeq {}^v Z_{\mu'}, \quad \widetilde{\operatorname{Ind}}(^v Z_{\mu'}) = {}^v Z_{\mu}.$$

This implies that we have $r = 0$.

Now we prove the statement of the lemma. The module $\tilde{E}_k(\Delta^{w(1_{\mu'})})$ has a graded Δ -filtration because it has a Δ -filtration as an ungraded module, see [13, Rem. 2.13]. The constituents (up to graded shifts) are $\Delta^{wz(1_{\mu})}$, $z \in W_{\mu'}/W_{\mu}$ by Lemma 4.18. We need only to identify the shifts. The graded multiplicities of Verma modules in projective modules are given in terms of Kazhdan-Lusztig polynomials in [13, App. A]. In particular, [13, Lem. A.4 (d)] implies that, for each $w \in {}^v J_{\mu}$, the module $\Delta^{w(1_{\mu})}$ appears as a constituent in a graded Δ -filtration of ${}^v P^{\mu}$ once with the graded shift by $\ell(w)$. Similarly, for each $w \in {}^v J_{\mu'}$, the module $\Delta^{w(1_{\mu'})}$ appears as a constituent in a graded Δ -filtration of ${}^v P^{\mu'}$ once with the graded shift by $\ell(w)$. Now, since $\tilde{E}_k({}^v P^{\mu'}) \simeq {}^v P^{\mu}$, we see that, for each $w \in {}^v J_{\mu'}$ and each $z \in J_{\mu'}^{\mu}$, the module $\Delta^{wz(1_{\mu})}$ appears in the Δ -filtration of $\tilde{E}_k(\Delta^{w(1_{\mu'})})$ with the graded shift by $\ell(z)$. \square

In the following lemma we consider the general (i.e., parabolic) situation.

Lemma 5.17. *For each $w \in {}^v J_{\mu'}^{\nu}$, we have $\tilde{E}_k({}^v P^{w(1_{\mu'})}) = {}^v P^{w(1_{\mu'})}$.*

Proof. By Lemmas 4.1 and 5.15, we have $\tilde{E}_k({}^v P^{w(1_{\mu'})}) = {}^v P^{w(1_{\mu'})}[r]$ for some integer r . We must show that the shift r is zero.

First, we prove this in the non-parabolic case. The module $\Delta^{w(1'_{\mu})}$ (resp. $\Delta^{w(1_{\mu})}$) is contained in each Δ -filtration of ${}^v P^{w(1_{\mu'})}$ (resp. ${}^v P^{w(1_{\mu})}$) only once and without a graded shift. Moreover, by Lemma 5.16 the module $\Delta^{w(1_{\mu})}$ is contained in each Δ -filtration of $\tilde{E}_k(\Delta^{w(1'_{\mu})})$ only once and without a graded shift. This implies that the graded shift r is zero.

The parabolic case follows from the non-parabolic case. Really, the projective covers of simple modules in the parabolic category \mathcal{O} are quotients of projective covers in the non-parabolic category \mathcal{O} (see Lemma 4.13 (b)). Thus the shift r should be zero in the parabolic case because it is zero in the non-parabolic case. \square

Let us check that the functor $\tilde{E}_k: {}^v \tilde{\mathcal{O}}_{\mu}^{\nu} \rightarrow {}^v \tilde{\mathcal{O}}_{\mu'}^{\nu}$ satisfies the hypotheses of Corollary 5.14. Condition (a) follows from Lemma 5.17.

Let P and Q be projective covers of simple modules in ${}^v \tilde{\mathcal{O}}_{\mu}$ graded as above. To check (b), we have to show that we have an isomorphism

$$\mathrm{Hom}(\tilde{E}_k(P)\langle 1 \rangle, \tilde{E}_k(Q)) \simeq \mathrm{Hom}(P\langle 1 \rangle, Q).$$

We have

$$\begin{aligned} \mathrm{Hom}(\tilde{E}_k(P)\langle 1 \rangle, \tilde{E}_k(Q)) &\simeq \mathrm{Hom}(P, \tilde{F}_{k+1} \tilde{E}_{k+1}(Q)\langle \mu_{k+1} - 1 \rangle) \\ &\simeq \mathrm{Hom}(P, [\mu_{k+1} + 1]_q(Q)\langle \mu_{k+1} - 1 \rangle) \\ &\simeq \mathrm{Hom}(P, Q\langle -1 \rangle) \oplus \bigoplus_{r=1}^{\mu_{k+1}} \mathrm{Hom}(P, Q\langle 2r - 1 \rangle) \\ &\simeq \mathrm{Hom}(P\langle 1 \rangle, Q). \end{aligned}$$

Here the first isomorphism follows from Lemma 4.29 (b), the second isomorphism follows from Lemma 4.31. The last isomorphism holds because $\mathrm{Hom}(P, Q\langle r \rangle)$ is

zero for $r > 0$ because the \mathbb{Z} -graded algebra

$$\text{End}\left(\bigoplus_{w \in {}^v J_\mu^\nu} {}^v P^{w(1_{\mu'})}\right)$$

has zero negative homogeneous components (as it is Koszul).

For each $\mu = (\mu_1, \dots, \mu_e)$ we set $\mu^{\text{op}} = (\mu_e, \dots, \mu_1)$. We can define the positive level version $O_{\mu,+}^\nu$ of the category O_μ^ν in the following way. For each $\lambda \in P$ we set $\tilde{\lambda}^+ = \lambda + z_\lambda \delta + (e - N)\Lambda_0$, where $z_\lambda = (\lambda, 2\rho + \lambda)/2e$. For each $\lambda \in P^\nu$ denote by ${}^+ \Delta(\lambda)$ the Verma module with highest weight $\tilde{\lambda}^+$ and denote by ${}^+ L(\lambda)$ its simple quotient. We will also abbreviate ${}^+ \Delta^\lambda = {}^+ \Delta(\lambda - \rho)$ and ${}^+ L^\lambda = {}^+ L(\lambda - \rho)$. Let $O_{\mu,+}^\nu$ be the Serre subcategory of O^ν generated by the simple modules ${}^+ L^\lambda$ for $\lambda \in P^\nu[\mu^{\text{op}}]$. Similarly to the negative e -action of \widehat{W} on P described in Section 3.2 we can consider the positive e -action on P . We define the positive e -action in the following way: the element $w \in \widehat{W}$ sends λ to $-w(-\lambda)$ (where $w(-\lambda)$ corresponds to the negative e -action). The notion of the positive e -action of \widehat{W} on P is motivated by the fact that the map

$$P \rightarrow \widehat{\mathbf{h}}^*, \quad \lambda \mapsto \widetilde{\lambda - \rho}^+ + \widehat{\rho}$$

is \widehat{W} -invariant. We say that an element $\lambda \in P$ is e -dominant if we have $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq \lambda_1 - e$. Fix an e -dominant element $1_\mu^+ \in P[\mu^{\text{op}}]$. (We can take for example $1_\mu^+ = (e^{\mu_1}, \dots, 1^{\mu_e})$). Note that the stabilizer of 1_μ^+ in \widehat{W} with respect to the positive e -action is W_μ . From now on, each time when we write $w(1_\mu^+)$ we mean the positive e -action on P and each time when we write $w(1_\mu)$ we mean the negative e -action.

Recall that $J_{\mu,+}$ is the subset of \widehat{W} containing all w such that w is maximal in wW_μ . Set $J_{\mu,+}^\nu = \{w \in J_{\mu,+}; w(1_\mu^+) \in P^\nu\}$. Note that the inclusion $J_{\mu'} \subset J_\mu$ induces an inclusion $J_{\mu'}^\nu \subset J_\mu^\nu$. For $v \in \widehat{W}$ we set ${}^v J_\mu^\nu = \{w \in J_\mu^\nu; w \leq v\}$ and ${}^v J_{\mu,+}^\nu = \{w \in J_{\mu,+}^\nu; w \leq v\}$.

We have the following lemma.

Lemma 5.18. (a) *There is a bijection $J_\mu^\nu \rightarrow J_{\nu,+}^\mu$ given by $w \mapsto w^{-1}$.*

(b) *For each $v \in J_\mu^\nu$, there is a bijection ${}^v J_\mu^\nu \rightarrow {}^{v^{-1}} J_{\nu,+}^\mu$ given by $w \mapsto w^{-1}$.*

Proof. Part (a) follows from [21, Cor. 3.3]. Part (b) follows from part (a). \square

Similarly to the truncated version ${}^v O_\mu^\nu$ of O_μ^ν , we can define the truncated version ${}^v O_{\mu,+}^\nu$ of $O_{\mu,+}^\nu$. We define ${}^v O_{\mu,+}^\nu$ as the Serre quotient of $O_{\mu,+}^\nu$, where we kill the simple module ${}^+ L^{w(1_\mu^+)}$ for each $w \in J_{\mu,+}^\nu - {}^v J_{\mu,+}^\nu$.

By [21, Thm. 3.12], for $v \in J_\mu^\nu$, the category ${}^v \widetilde{O}_\mu^\nu$ is Koszul dual to the category ${}^{v^{-1}} \widetilde{O}_{\nu,+}^\mu$. The bijection between the simple modules in ${}^v \widetilde{O}_\mu^\nu$ and the indecomposable projective modules in ${}^{v^{-1}} \widetilde{O}_{\nu,+}^\mu$ given by the Koszul functor \mathcal{K} is such that for each $w \in {}^v J_\mu^\nu$ the module $L^{w(1_\mu)}$ corresponds to the projective cover of ${}^+ L^{w^{-1}(1_\nu^+)}$.

We should make a remark about our notation. Usually, we denote by e the number of components in μ and we denote by l the number of components in ν . So, when we exchange the roles of μ and ν and we consider the category $O_{\nu,+}^\mu$, we mean that this category is defined with respect to the level $l - N$ (and not $e - N$).

Now, assume again that v is in $J_{\mu'}^\nu w_{\mu'}$. Then we have $vw_\mu \in J_\mu^\nu$ and $vw_{\mu'} \in J_{\mu'}^\nu$. In this case the Koszul dual categories to ${}^v O_\mu^\nu$ and ${}^v O_{\mu'}^\nu$ are ${}^{w_\mu v^{-1}} O_{\nu,+}^\mu$ and ${}^{w_{\mu'} v^{-1}} O_{\nu,+}^{\mu'}$.

Lemma 5.19. (a) *We have*

$${}^{w_{\mu'} v^{-1}} J_{\nu,+}^{\mu'} = {}^{w_\mu v^{-1}} J_{\nu,+}^\mu \cap J_{\nu,+}^{\mu'}.$$

(b) *We have*

$${}^{w_{\mu'} v^{-1}} J_{\nu,+}^{\mu'} = {}^{w_\mu v^{-1}} J_{\nu,+}^{\mu'}.$$

Proof. Let us prove (a). By Lemma 5.18 the statement is equivalent to

$${}^{vw_{\mu'}} J_{\mu'}^\nu = {}^{vw_\mu} J_\mu^\nu \cap J_{\mu'}^\nu.$$

Moreover, by definition, we have ${}^{vw_{\mu'}} J_{\mu'}^\nu = {}^v J_{\mu'}^\nu$ and ${}^{vw_\mu} J_\mu^\nu = {}^v J_\mu^\nu$. Thus, the statement is equivalent to ${}^v J_{\mu'}^\nu = {}^v J_\mu^\nu \cap J_{\mu'}^\nu$. The last equality is obvious.

Part (b) follows from part (a). □

Now, put $u = w_\mu v^{-1}$. The discussion above together with Lemma 5.19 shows that the Koszul dual categories to ${}^v O_\mu^\nu$ and ${}^v O_{\mu'}^\nu$ are ${}^u O_{\nu,+}^\mu$ and ${}^u O_{\nu,+}^{\mu'}$.

We get the following result.

Theorem 5.20. *Assume that we have $W_\mu \subset W_{\mu'}$.*

(a) *The functor $\tilde{F}_k: D^b({}^v \tilde{O}_\mu^\nu) \rightarrow D^b({}^v \tilde{O}_{\mu'}^\nu)$ is Koszul dual to the shifted parabolic truncation functor $\tilde{\text{tr}}\langle \mu_{k+1} \rangle: D^b({}^u \tilde{O}_{\nu,+}^\mu) \rightarrow D^b({}^u \tilde{O}_{\nu,+}^{\mu'})$.*

(b) *The functor $\tilde{E}_k: D^b({}^v \tilde{O}_{\mu'}^\nu) \rightarrow D^b({}^v \tilde{O}_\mu^\nu)$ is Koszul dual to the parabolic inclusion functor $\tilde{\text{inc}}: D^b({}^u \tilde{O}_{\nu,+}^{\mu'}) \rightarrow D^b({}^u \tilde{O}_{\nu,+}^\mu)$.*

Proof. We have checked above that the functor $\tilde{E}_k: {}^v \tilde{O}_\mu^\nu \rightarrow {}^v \tilde{O}_{\mu'}^\nu$ satisfies the hypotheses of Corollary 5.14. Thus Corollary 5.14 implies part (b). Part (a) follows from part (b) by adjointness. □

Similarly to the situation $W_\mu \subset W_{\mu'}$, we can do the same in the situation $W_\mu \subset W_{\mu'}$ (see also Section 4.9). In this case we should take $v \in J_\mu^\nu w_\mu$ and put $u = w_{\mu'} v^{-1}$. We get the following theorem.

Theorem 5.21. *Assume that we have $W_{\mu'} \subset W_\mu$.*

(a) *The functor $\tilde{F}_k: D^b({}^v \tilde{O}_\mu^\nu) \rightarrow D^b({}^v \tilde{O}_{\mu'}^\nu)$ is Koszul dual to the parabolic inclusion functor $\tilde{\text{inc}}: D^b({}^u \tilde{O}_{\nu,+}^\mu) \rightarrow D^b({}^u \tilde{O}_{\nu,+}^{\mu'})$.*

(b) *The functor $\tilde{E}_k: D^b({}^v \tilde{O}_{\mu'}^\nu) \rightarrow D^b({}^v \tilde{O}_\mu^\nu)$ is Koszul dual to the shifted parabolic truncation functor $\tilde{\text{tr}}\langle \mu_k - 1 \rangle: D^b({}^u \tilde{O}_{\nu,+}^{\mu'}) \rightarrow D^b({}^u \tilde{O}_{\nu,+}^\mu)$.* □

5.8 The restriction to the category \mathbf{A}

The goal of this section is to restrict the results of the previous section to category \mathbf{A} .

We have seen that we can grade the functor E_k and F_k for category \mathcal{O} when we have $W_\mu \subset W_{\mu'}$ or $W_{\mu'} \subset W_\mu$. Let us show that in this cases we can also grade similar functors for the category \mathbf{A} . We have $\mathbf{A}^\nu[\alpha] \subset {}^\nu O_\mu^\nu$ and $\mathbf{A}^\nu[\alpha + \alpha_k] \subset {}^\nu O_{\mu'}^\nu$. Denote by h the inclusion functor from $\mathbf{A}^\nu[\alpha]$ to ${}^\nu O_\mu^\nu$. Abusing the notation, we will use the same symbol for the inclusion functor from $\mathbf{A}^\nu[\alpha + \alpha_k]$ to ${}^\nu O_{\mu'}^\nu$. Let h^* and $h^!$ be the left and right adjoint functors to h . The functor F_k for the category \mathbf{A} is defined as the restriction of the functor F_k for the category \mathcal{O} . This restriction can be written as $h^! F_k h$. The functor E_k for the category \mathcal{O} does not preserve the category \mathbf{A} in general. The functor E_k for the category \mathbf{A} is defined in [19, Sec. 5.9] as $h^* E_k h$. It is easy to see that we can grade the functor h and its adjoint functors in the same way as we graded Zuckerman functors. Thus we obtain graded lifts \tilde{E}_k and \tilde{F}_k of the functors E_k and F_k for the category \mathbf{A} . Moreover, we still have the adjunctions $(\tilde{E}_k, \tilde{F}_k \langle \mu_{k+1} \rangle)$ (when $W_\mu \subset W_{\mu'}$) and $(\tilde{E}_k, \tilde{F}_k \langle 1 - \mu_k \rangle)$ (when $W_{\mu'} \subset W_\mu$) in the category \mathbf{A} .

We do not have adjunctions in other direction in general. However, if additionally we have $\nu_r > |\alpha|$ for each $r \in [1, l]$, then the functors E_k and F_k for the category \mathbf{A} are biadjoint by [19, Lem. 7.6]. This means that there is no difference between $h^* E_k h$ and $h^! E_k h$. Thus we also get the adjunctions $(\tilde{F}_k, \tilde{E}_k \langle -\mu_{k+1} \rangle)$ (when $W_\mu \subset W_{\mu'}$) and $(\tilde{F}_k, \tilde{E}_k \langle \mu_k - 1 \rangle)$ (when $W_{\mu'} \subset W_\mu$) in the category \mathbf{A} . (In fact, we always have the adjunctions in both directions if $k \neq 0$ because in this case the functor E_k for the category \mathbf{A} is just the restriction of the functor E_k for the category \mathcal{O} and similarly for \tilde{E}_k .)

We start from a general lemma. Let A be a finite dimensional Koszul algebra over \mathbb{C} . Let $\{e_\lambda; \lambda \in \Lambda\}$ be the set of indecomposable idempotents in A . Fix a subset $\Lambda' \subset \Lambda$. Assume that the algebra ${}_{\Lambda'} A$ (see Section 5.5 for the notations) is also Koszul. Then we have an algebra isomorphism $({}_{\Lambda'} A)^! \simeq (A^!)_{\Lambda'}$. The graded algebra ${}_{\Lambda'} A$ is a quotient of the graded algebra A by a homogeneous ideal. In particular we have an inclusion of categories $\iota: \text{grmod}({}_{\Lambda'} A) \rightarrow \text{grmod}(A)$. Moreover, there is a functor

$$\tau: \text{grmod}(A^!) \rightarrow \text{grmod}((A^!)_{\Lambda'}), \quad M \mapsto e_{\Lambda'}^! M.$$

The functors ι and τ are both exact. They yield functors between derived categories $\iota: D^b({}_{\Lambda'} A) \rightarrow D^b(A)$ and $\tau: D^b(A^!) \rightarrow D^b((A^!)_{\Lambda'})$.

Since the algebra A is Koszul, there is a functor $\mathcal{K}: D^b(A) \rightarrow D^b(A^!)$ defined by $\mathcal{K} = \text{RHom}(A_0, \bullet)$, see Section 5.3. We will sometimes write \mathcal{K}_A to specify the algebra A .

In the following lemma we identify $({}_{\Lambda'} A)^! = (A^!)_{\Lambda'}$.

Lemma 5.22. *We have the following isomorphism of functors $D^b({}_{\Lambda'} A) \rightarrow D^b((A^!)_{\Lambda'})$*

$$\mathcal{K}_{\Lambda' A} \simeq \tau \circ \mathcal{K}_A \circ \iota.$$

Proof. For a complex $M \in D^b(\Lambda' A)$, we have

$$\begin{aligned} \tau \circ \mathcal{K}_A \circ i(M) &\simeq \tau(\mathrm{RHom}_A(A_0, M)) \\ &\simeq \mathrm{RHom}_A(e_{\Lambda'} A_0, M) \\ &\simeq \mathrm{RHom}_{\Lambda' A}((\Lambda' A)_0, M) \\ &\simeq \mathcal{K}_{\Lambda' A}(M). \end{aligned}$$

□

Fix $\alpha \in Q_I^+$. Consider the category $\mathbf{A}^\nu[\alpha]$ as in Section 3.9. Let μ be such that $\mathbf{A}^\nu[\alpha]$ is a subcategory of O_μ^ν . (Then $\mathbf{A}^\nu[\alpha + \alpha_k]$ is a subcategory of $O_{\mu'}^\nu$.) Assume that we have $W_\mu \subset W_{\mu'}$. Assume that $v \in J_{\mu'}^\nu w_{\mu'}$ is such that $\mathbf{A}^\nu[\alpha]$ is a subcategory of ${}^v O_\mu^\nu$ and $\mathbf{A}^\nu[\alpha + \alpha_k]$ is a subcategory of ${}^v O_{\mu'}^\nu$. Put $u = w_\mu v^{-1}$. The category $\mathbf{A}^\nu[\alpha]$ is also Koszul. Denote by $\tilde{\mathbf{A}}^\nu[\alpha]$ its graded version. The Koszul dual category to $\mathbf{A}^\nu[\alpha]$ is a Serre quotient of the category ${}^u O_{\nu,+}^\mu$ (see [13, Rem. 3.15]). Let us denote this quotient and its graded version by $\mathbf{A}_+^\mu[\alpha]$ and $\tilde{\mathbf{A}}_+^\mu[\alpha]$ respectively. (We will also use similar notations for $\mathbf{A}^\nu[\alpha + \alpha_k]$.)

First, we prove the following lemma.

Lemma 5.23. *Assume that we have $W_\mu \subset W_{\mu'}$ and $k \neq 0$.*

- (a) *The inclusion of categories ${}^u O_{\nu,+}^{\mu'} \subset {}^u O_{\nu,+}^\mu$ yields an inclusion of categories $\mathbf{A}_+^{\mu'}[\alpha + \alpha_k] \subset \mathbf{A}_+^\mu[\alpha]$.*
- (b) *The inclusion of categories ${}^u \tilde{O}_{\nu,+}^{\mu'} \subset {}^u \tilde{O}_{\nu,+}^\mu$ yields an inclusion of categories $\tilde{\mathbf{A}}_+^{\mu'}[\alpha + \alpha_k] \subset \tilde{\mathbf{A}}_+^\mu[\alpha]$.*
Assume that we have $W_\mu \supset W_{\mu'}$ and $k \neq 0$.
- (c) *The inclusion of categories ${}^u O_{\nu,+}^\mu \subset {}^u O_{\nu,+}^{\mu'}$ yields an inclusion of categories $\mathbf{A}_+^\mu[\alpha] \subset \mathbf{A}_+^{\mu'}[\alpha + \alpha_k]$.*
- (d) *The inclusion of categories ${}^u \tilde{O}_{\nu,+}^\mu \subset {}^u \tilde{O}_{\nu,+}^{\mu'}$ yields an inclusion of categories $\tilde{\mathbf{A}}_+^\mu[\alpha] \subset \tilde{\mathbf{A}}_+^{\mu'}[\alpha + \alpha_k]$.*

Proof. Denote by p_1 and p_2 respectively the quotient functors

$$p_1: {}^u O_{\nu,+}^{\mu'} \rightarrow \mathbf{A}_+^{\mu'}[\alpha + \alpha_k], \quad p_2: {}^u O_{\nu,+}^\mu \rightarrow \mathbf{A}_+^\mu[\alpha].$$

To prove (a) and (b), it is enough to prove that each simple module in ${}^u O_{\nu,+}^{\mu'}$ is killed by the functor p_1 if and only if it is killed by the functor p_2 . We can get the combinatorial description of the simple modules killed by p_1 and p_2 respectively using [13, Rem. 2.18].

For each $w \in {}^v J_{\mu'}^\nu$ (resp. $w \in {}^v J_\mu^\nu$), the simple module ${}^+ L^{w^{-1}(1_\nu^+)}$ is killed by p_1 (resp. p_2) if and only if the simple module $L^{w(1_{\mu'})} \in {}^v O_{\mu'}^\nu$ is not in $\mathbf{A}^\nu[\alpha + \alpha_k]$ (resp. the simple module $L^{w(1_\mu)} \in {}^v O_\mu^\nu$ is not in $\mathbf{A}^\nu[\alpha]$). So, we need to show that for each $w \in {}^v J_{\mu'}^\nu$ the module $L^{w(1_{\mu'})} \in {}^v O_{\mu'}^\nu$ is in $\mathbf{A}^\nu[\alpha + \alpha_k]$ if and only if the module $L^{w(1_\mu)} \in {}^v O_\mu^\nu$ is in $\mathbf{A}^\nu[\alpha]$. Finally, we have to show that for each

$w \in {}^v J_{\mu'}^\nu$ we have $w(1_{\mu'}) \geq \rho_\nu$ if and only if we have $w(1_\mu) \geq \rho_\nu$. (Here the order is as in Section 3.10.)

It is obvious that $w(1_\mu) \geq \rho_\nu$ implies $w(1_{\mu'}) \geq \rho_\nu$ because we have $w(1_{\mu'}) \geq w(1_\mu)$. Now, let us show the inverse statement. Note that we have $w(1_{\mu'}) = w(1_\mu) + \epsilon_r$, where $r \in [1, N]$ is the unique index such that $w(1_\mu)_r \equiv k \pmod{e}$. Assume that we have $w(1_{\mu'}) \geq \rho_\nu$ but not $w(1_\mu) \geq \rho_\nu$. Then we have $w(1_{\mu'})_r = (\rho_\nu)_r$. Assume first that $(\rho_\nu)_r \neq 1$. In particular this implies $r < N$. Since the weight $w(1_\mu)$ is in P^ν , we have

$$w(1_{\mu'})_{r+1} = w(1_\mu)_{r+1} < w(1_\mu)_r = (\rho_\nu)_r - 1 = (\rho_\nu)_{r+1}.$$

This contradicts to $w(1_{\mu'}) \geq \rho_\nu$. Now, assume that we have $(\rho_\nu)_r = 1$. Since we have $(\rho_\nu)_r \equiv w(1_{\mu'})_r \equiv k + 1 \pmod{e}$, this implies $k = 0$. This contradicts with the assumption $k \neq 0$. This proves the statement.

The proof of (c), (d) is similar to the proof of (a) and (b). \square

In the case $W_\mu \subset W_{\mu'}$, $k \neq 0$, the lemma above allows us to define the parabolic inclusion functor $\text{inc}: \mathbf{A}_+^{\mu'}[\alpha + \alpha_k] \rightarrow \mathbf{A}_+^\mu[\alpha]$ and the parabolic truncation functor $\text{tr}: \mathbf{A}_+^\mu[\alpha] \rightarrow \mathbf{A}_+^{\mu'}[\alpha + \alpha_k]$ and their graded versions $\widetilde{\text{inc}}$ and $\widetilde{\text{tr}}$. Similarly, in the case $W_\mu \supset W_{\mu'}$, $k \neq 0$, the lemma above allows us to define the parabolic inclusion functor $\text{inc}: \mathbf{A}_+^\mu[\alpha] \rightarrow \mathbf{A}_+^{\mu'}[\alpha + \alpha_k]$ and the parabolic truncation functor $\text{tr}: \mathbf{A}_+^{\mu'}[\alpha + \alpha_k] \rightarrow \mathbf{A}_+^\mu[\alpha]$ and their graded versions $\widetilde{\text{inc}}$ and $\widetilde{\text{tr}}$.

Theorem 5.24. *Assume that we have $W_\mu \subset W_{\mu'}$.*

(a) *The functor $\widetilde{F}_k: D^b(\widetilde{\mathbf{A}}^\nu[\alpha]) \rightarrow D^b(\widetilde{\mathbf{A}}^\nu[\alpha + \alpha_k])$ is Koszul dual to the shifted parabolic truncation functor $\widetilde{\text{tr}}_{\langle \mu_{k+1} \rangle}: D^b(\widetilde{\mathbf{A}}_+^\mu[\alpha]) \rightarrow D^b(\widetilde{\mathbf{A}}_+^{\mu'}[\alpha + \alpha_k])$.*

(b) *The functor $\widetilde{E}_k: D^b(\widetilde{\mathbf{A}}^\nu[\alpha + \alpha_k]) \rightarrow D^b(\widetilde{\mathbf{A}}^\nu[\alpha])$ is Koszul dual to the parabolic inclusion functor $\widetilde{\text{inc}}: D^b(\widetilde{\mathbf{A}}_+^{\mu'}[\alpha + \alpha_k]) \rightarrow D^b(\widetilde{\mathbf{A}}_+^\mu[\alpha])$.*

Now, assume that we have $W_{\mu'} \subset W_\mu$.

(c) *The functor $\widetilde{F}_k: D^b(\widetilde{\mathbf{A}}^\nu[\alpha]) \rightarrow D^b(\widetilde{\mathbf{A}}^\nu[\alpha + \alpha_k])$ is Koszul dual to the parabolic inclusion functor $\widetilde{\text{inc}}: D^b(\widetilde{\mathbf{A}}_+^\mu[\alpha]) \rightarrow D^b(\widetilde{\mathbf{A}}_+^{\mu'}[\alpha + \alpha_k])$.*

(d) *The functor $\widetilde{E}_k: D^b(\widetilde{\mathbf{A}}^\nu[\alpha + \alpha_k]) \rightarrow D^b(\widetilde{\mathbf{A}}^\nu[\alpha])$ is Koszul dual to the shifted parabolic truncation functor $\widetilde{\text{tr}}_{\langle \mu_k - 1 \rangle}: D^b(\widetilde{\mathbf{A}}_+^{\mu'}[\alpha + \alpha_k]) \rightarrow D^b(\widetilde{\mathbf{A}}_+^\mu[\alpha])$.*

Proof. Let us prove (b).

Let $v \in J_{\mu'}^\nu w_{\mu'}$ be such that $\mathbf{A}^\nu[\alpha]$ is a subcategory of ${}^v O_{\mu'}^\nu$ and $\mathbf{A}^\nu[\alpha + \alpha_k]$ is a subcategory of ${}^v O_{\mu'}^\nu$. Then the same is true for graded versions. Denote by i the inclusion functor from $\widetilde{\mathbf{A}}^\nu[\alpha]$ to ${}^v \widetilde{O}_{\mu'}^\nu$. Let $\tau: {}^u \widetilde{O}_{\nu,+}^\mu \rightarrow \widetilde{\mathbf{A}}_+^\mu[\alpha]$ be the natural quotient functor.

Consider the following diagram

$$\begin{array}{ccc}
D^b(\tilde{\mathbf{A}}_+^\mu[\alpha]) & \xleftarrow{\widetilde{\text{inc}}} & D^b(\tilde{\mathbf{A}}_+^{\mu'}[\alpha + \alpha_k]) \\
\tau \uparrow & & \tau \uparrow \\
D^b({}^v\tilde{O}_{\nu,+}^\mu) & \xleftarrow{\widetilde{\text{inc}}} & D^b({}^v\tilde{O}_{\nu,+}^{\mu'}) \\
\kappa \uparrow & & \kappa \uparrow \\
D^b({}^v\tilde{O}_\mu^\nu) & \xleftarrow{E_k} & D^b({}^v\tilde{O}_{\mu'}^\nu) \\
i \uparrow & & i \uparrow \\
D^b(\tilde{\mathbf{A}}^\nu[\alpha]) & \xleftarrow{E_k} & D^b(\tilde{\mathbf{A}}^\nu[\alpha + \alpha_k]).
\end{array}$$

The commutativity of the top and bottom rectangles is obvious. The commutativity of the middle rectangle follows from Theorem 5.20 (b). Now, by Lemma 5.22, the big rectangle in the diagram above yields the following commutative diagram

$$\begin{array}{ccc}
D^b(\tilde{\mathbf{A}}_+^\mu[\alpha]) & \xleftarrow{\widetilde{\text{inc}}} & D^b(\tilde{\mathbf{A}}_+^{\mu'}[\alpha + \alpha_k]) \\
\kappa \uparrow & & \kappa \uparrow \\
D^b(\tilde{\mathbf{A}}^\nu[\alpha]) & \xleftarrow{F_k} & D^b(\tilde{\mathbf{A}}^\nu[\alpha + \alpha_k]).
\end{array}$$

This proves (b).

Part (a) follows from (b) by adjointness. We can prove (c) in the same way as (b), using Theorem 5.21 (a). Part (d) follows from (c) by adjointness. \square

5.9 Zuckerman functors for the category \mathbf{A}_+

Fix $u \in \widehat{W}$. The Zuckerman functor $\text{Zuc}: {}^uO_{\nu,+}^\mu \rightarrow {}^uO_{\nu,+}^{\mu'}$ (see Section 5.6) is a composition of a parabolic inclusion functor and a parabolic truncation functor ${}^uO_{\nu,+}^\mu \xrightarrow{\text{inc}} {}^uO_{\nu,+}^{\mu''} \xrightarrow{\text{tr}} {}^uO_{\nu,+}^{\mu'}$, where the parabolic type μ'' is chosen such that $W_{\mu''} = W_\mu \cap W_{\mu'}$ (in fact, we can take $\mu'' = \overline{\mu}^0$). Now we are going to give an analogue of the Zuckerman functor for the category \mathbf{A}_+ , i.e., we want to define a functor $\text{Zuc}_k^+: \mathbf{A}_+^\mu[\alpha] \rightarrow \mathbf{A}_+^{\mu'}[\alpha + \alpha_k]$. (Recall that the categories $\mathbf{A}_+^\mu[\alpha]$ and $\mathbf{A}_+^{\mu'}[\alpha + \alpha_k]$ are Serre quotients of ${}^uO_{\nu,+}^\mu$ and ${}^uO_{\nu,+}^{\mu'}$ respectively for u big enough.) The main difficulty to give such a definition is that we have no obvious candidate to replace the category ${}^uO_{\nu,+}^{\mu''}$.

Let us write $\overline{\mathbf{A}}$ instead of \mathbf{A} to indicate that the category is defined with respect to $e+1$ instead of e . Let us identify $\mathbf{A}_+^\mu[\alpha] \simeq \overline{\mathbf{A}}_+^{\overline{\mu}}[\beta + \overline{\alpha}]$ and $\mathbf{A}_+^{\mu'}[\alpha + \alpha_k] \simeq \overline{\mathbf{A}}_+^{\overline{\mu}'}[\beta + \overline{\alpha} + \overline{\alpha}_k + \overline{\alpha}_{k+1}]$ (see Proposition 3.18). Assume that we have $k \neq 0$. Then by Lemma 5.23 we have the following inclusion of categories

$$\overline{\mathbf{A}}_+^{\overline{\mu}}[\beta + \overline{\alpha}] \subset \overline{\mathbf{A}}_+^{\overline{\mu}^0}[\beta + \overline{\alpha} + \overline{\alpha}_k] \supset \overline{\mathbf{A}}_+^{\overline{\mu}'}[\beta + \overline{\alpha} + \overline{\alpha}_k + \overline{\alpha}_{k+1}].$$

Now, we define the Zuckerman functor $\text{Zuc}_k^+ : \mathbf{A}_+^\mu[\alpha] \rightarrow \mathbf{A}_+^{\mu'}[\alpha + \alpha_k]$ as the composition of the parabolic inclusion functor with the parabolic truncation functor

$$\mathbf{A}_+^\mu[\alpha] \xrightarrow{\text{inc}} \overline{\mathbf{A}}_+^{\mu^0}[\beta + \overline{\alpha} + \overline{\alpha}_k] \xrightarrow{\text{tr}} \mathbf{A}_+^{\mu'}[\alpha + \alpha_k].$$

We define the Zuckerman functor $\text{Zuc}_k^- : \mathbf{A}_+^{\mu'}[\alpha + \alpha_k] \rightarrow \mathbf{A}_+^\mu[\alpha]$ in a similar way.

We can also define the graded version $\widetilde{\text{Zuc}}_k^\pm$ of the Zuckerman functors by replacing the functors inc and tr by their graded versions $\widetilde{\text{inc}}$ and $\widetilde{\text{tr}}$. Unfortunately this approach does not allow to define the Zuckerman functors for $k = 0$ because of the assumption $k \neq 0$ in Lemma 5.23. The definition of the Zuckerman functors for $k = 0$ will be given in Section 5.11.

5.10 The Koszul dual functors in the category \mathbf{A}

Theorem 5.25. *Assume that we have $\nu_r > |\alpha|$ for each $r \in [1, l]$, $e > 2$ and $k \neq 0$.*

(a) *The functor $F_k : \mathbf{A}^\nu[\alpha] \rightarrow \mathbf{A}^\nu[\alpha + \alpha_k]$ has a graded lift \widetilde{F}_k such that the functor $\widetilde{F}_k : D^b(\widetilde{\mathbf{A}}^\nu[\alpha]) \rightarrow D^b(\widetilde{\mathbf{A}}^\nu[\alpha + \alpha_k])$ is Koszul dual to the shifted Zuckerman functor $\widetilde{\text{Zuc}}_k^+(\mu_{k+1}) : D^b(\widetilde{\mathbf{A}}_+^\mu[\alpha]) \rightarrow D^b(\widetilde{\mathbf{A}}_+^{\mu'}[\alpha + \alpha_k])$.*

(b) *The functor $E_k : \mathbf{A}^\nu[\alpha + \alpha_k] \rightarrow \mathbf{A}^\nu[\alpha]$ has a graded lift such that the functor $\widetilde{E}_k : D^b(\widetilde{\mathbf{A}}^\nu[\alpha + \alpha_k]) \rightarrow D^b(\widetilde{\mathbf{A}}^\nu[\alpha])$ is Koszul dual to the shifted Zuckerman functor $\widetilde{\text{Zuc}}_k^-(\mu_k - 1) : D^b(\widetilde{\mathbf{A}}_+^{\mu'}[\alpha]) \rightarrow D^b(\widetilde{\mathbf{A}}_+^\mu[\alpha + \alpha_k])$.*

Proof. By Theorem 3.44 we have the following commutative diagram

$$\begin{array}{ccccc} \overline{\mathbf{A}}^\nu[\beta + \overline{\alpha}] & \xrightarrow{\widetilde{F}_k} & \overline{\mathbf{A}}^\nu[\beta + \overline{\alpha} + \overline{\alpha}_k] & \xrightarrow{\widetilde{F}_{k+1}} & \overline{\mathbf{A}}^\nu[\beta + \overline{\alpha} + \overline{\alpha}_k + \overline{\alpha}_{k+1}] \\ \uparrow & & & & \downarrow \\ \mathbf{A}^\nu[\alpha] & \xrightarrow{F_k} & & & \mathbf{A}^\nu[\alpha + \alpha_k] \end{array}$$

Here the vertical maps are some equivalences of categories. By unicity of Koszul grading (see [3, Cor. 2.5.2]) there exist unique graded lifts of vertical maps such that they are equivalences of graded categories and they respect the chosen grading of simple modules (i.e., concentrated in degree 0). Moreover, the top horizontal maps have graded lifts because for a suitable v we have

$$\overline{\mathbf{A}}^\nu[\beta + \overline{\alpha}] \subset {}^v\overline{\mathcal{O}}_{\overline{\mu}}, \quad \overline{\mathbf{A}}^\nu[\beta + \overline{\alpha} + \overline{\alpha}_k] \subset {}^v\overline{\mathcal{O}}_{\overline{\mu}^0}, \quad \overline{\mathbf{A}}^\nu[\beta + \overline{\alpha} + \overline{\alpha}_k + \overline{\alpha}_{k+1}] \subset {}^v\overline{\mathcal{O}}_{\overline{\mu}'}$$

and $W_{\overline{\mu}} \supset W_{\overline{\mu}^0} \subset W_{\overline{\mu}'}$. This implies that there is a graded version \widetilde{F}_k of the functor F_k such that it makes the graded version of the diagram above commutative.

Since the categories $\mathbf{A}^\nu[\alpha]$ and $\overline{\mathbf{A}}^\nu[\beta + \overline{\alpha}]$ are equivalent, their Koszul dual categories are also equivalent. We can chose the equivalences $(\mathbf{A}^\nu[\alpha])^! \simeq \mathbf{A}_+^\mu[\alpha]$

and $(\bar{\mathbf{A}}^\nu[\beta + \bar{\alpha}])^! \simeq \mathbf{A}_+^\mu[\alpha]$ in such a way that the vertical map in the diagram is Koszul dual to the identity functor. We can do the same with the categories in the right part of the diagram above.

By Theorem 5.24, the left top functor in the graded version of the diagram above is Koszul dual to the parabolic inclusion functor inc and the top right functor in the diagram is Koszul dual to the graded shift $\text{tr}\langle\mu_{k+1}\rangle$ of the parabolic truncation functor. By definition (see Section 5.6), the Zuckerman functor is the composition of the parabolic inclusion and the parabolic truncation functors. This implies that the functor $\tilde{F}_k: D^b(\bar{\mathbf{A}}^\nu[\alpha]) \rightarrow D^b(\bar{\mathbf{A}}^\nu[\alpha + \alpha_k])$ is Koszul dual to the shifted Zuckerman functor $\widetilde{\text{Zuc}}_k^+ \langle\mu_{k+1}\rangle$. This proves (a).

We can prove (b) in the same way. By adjointness, the diagram above yields a similar diagram for the functor E . This diagram allows to grade the functor E_k . Then we deduce the Koszul dual functor to E_k in the same way as in (a). \square

5.11 The case $k = 0$

Now, we are going to get an analogue of Theorem 5.29 in the case $k = 0$. The main difficulty in this case is that we cannot define Zuckerman functors for the category \mathbf{A}_+ in the same way as in Section 5.9 because Lemma 5.23 fails. To fix this problem we replace the category \mathbf{A} by a smaller category A .

Assume that we have $k = 0$ and $W_\mu \supset W_{\mu'}$. In particular this implies $\mu_1 = 0$.

Let $A^\nu[\alpha + \alpha_0]$ be the Serre subcategory of $\mathbf{A}^\nu[\alpha + \alpha_0]$ generated by simple modules L^λ such that the weight $\lambda \in P$ has no coordinates equal to 1. It is a highest weight subcategory.

Remark 5.26. (a) The category $A^\nu[\alpha + \alpha_0]$ inherits the Koszul grading from the category $\mathbf{A}^\nu[\alpha + \alpha_0]$ in the following way. We know that there is a Koszul algebra A such that $\mathbf{A}^\nu[\alpha + \alpha_0] \simeq \text{mod}(A)$. Let $\{e_\lambda; \lambda \in \Lambda\}$ be the set of indecomposable idempotents of A_0 . Then by [13, Lem. 2.17] there is a subset $\Lambda' \subset \Lambda$ such that we have $\mathbf{A}^\nu[\alpha + \alpha_0] \simeq \text{mod}({}_{\Lambda'}A)$ (see Section 5.5 for the notations). Moreover, the Koszul dual algebra to ${}_{\Lambda'}A$ is $A_{\Lambda'}^!$.

Since, we have $\text{mod}(A^!) \simeq \mathbf{A}_+^{\mu'}[\alpha + \alpha_0]$, the Koszul dual category $A_+^{\mu'}[\alpha + \alpha_0]$ to $A^\nu[\alpha + \alpha_0]$ is a Serre quotient of $\mathbf{A}_+^{\mu'}[\alpha + \alpha_0]$. The quotient functor

$$a: \mathbf{A}_+^{\mu'}[\alpha + \alpha_0] \rightarrow A_+^{\mu'}[\alpha + \alpha_0]$$

can be seen as the functor

$$a: \text{mod}(A^!) \rightarrow \text{mod}(A_{\Lambda'}^!), \quad M \mapsto e_{\Lambda'}^! M.$$

(b) The left adjoint functor $b: A_+^{\mu'}[\alpha + \alpha_0] \rightarrow \mathbf{A}_+^{\mu'}[\alpha + \alpha_0]$ to a can be seen as

$$b: \text{mod}(A_{\Lambda'}^!) \rightarrow \text{mod}(A^!), \quad M \mapsto A^! e_{\Lambda'}^! \otimes_{A_{\Lambda'}^!} M.$$

The functors a and b have obvious graded lifts

$$\tilde{a}: \tilde{\mathbf{A}}_+^{\mu'}[\alpha + \alpha_0] \rightarrow \tilde{A}_+^{\mu'}[\alpha + \alpha_0], \quad \tilde{b}: \tilde{A}_+^{\mu'}[\alpha + \alpha_0] \rightarrow \tilde{\mathbf{A}}_+^{\mu'}[\alpha + \alpha_0].$$

By Proposition 5.11, the functor \tilde{b} is Koszul dual to the inclusion functor $\tilde{A}^\nu[\alpha + \alpha_0] \rightarrow \tilde{\mathbf{A}}^\nu[\alpha + \alpha_0]$. Then, by adjointness, the functor \tilde{a} is Koszul dual to the right adjoint functor to the inclusion functor above.

It is easy to see from the action of F_0 on Verma modules (see Proposition 3.10 (e)) that the image of the functor $F_0: \mathbf{A}^\nu[\alpha] \rightarrow \mathbf{A}^\nu[\alpha + \alpha_0]$ is in $A^\nu[\alpha + \alpha_0]$. Moreover, recall from Section 3.9 that the functor $E_0: O_{\mu'}^\nu \rightarrow O_\mu^\nu$ does not take $\mathbf{A}^\nu[\alpha + \alpha_0]$ to $\mathbf{A}^\nu[\alpha]$. (The reader should pay attention to the fact that the functor E_0 for the category \mathbf{A} is not defined as the restriction of the functor E_0 for the category O .) However, it is easy to see from the action of E_0 on Verma modules (see Proposition 3.10 (e)) that the functor E_0 for the category O takes $A^\nu[\alpha + \alpha_0]$ to $\mathbf{A}^\nu[\alpha]$. Thus we get a functor $E_0: A^\nu[\alpha + \alpha_0] \rightarrow \mathbf{A}^\nu[\alpha]$. This functor also coincides with the restriction of the functor $E_0: \mathbf{A}^\nu[\alpha + \alpha_0] \rightarrow \mathbf{A}^\nu[\alpha]$ to the category $A^\nu[\alpha + \alpha_0]$.

The following statement can be proved in the same way as Lemma 5.23.

Lemma 5.27. *Assume that we have $W_\mu \supset W_{\mu'}$.*

(a) *The inclusion of categories ${}^u O_{\nu,+}^\mu \subset {}^u O_{\nu,+}^{\mu'}$ yields an inclusion of categories $\mathbf{A}_+^\mu[\alpha] \subset A_+^{\mu'}[\alpha + \alpha_0]$.*

(b) *The inclusion of categories ${}^u \tilde{O}_{\nu,+}^{\mu'} \subset {}^u \tilde{O}_{\nu,+}^\mu$ yields an inclusion of categories $\tilde{\mathbf{A}}_+^{\mu'}[\alpha + \alpha_0] \subset \tilde{A}_+^\mu[\alpha]$.*

□

The lemma above allows us to define the inclusion and the truncation functors $\text{inc}: \mathbf{A}_+^\mu[\alpha] \rightarrow A_+^{\mu'}[\alpha + \alpha_0]$, $\text{tr}: A_+^{\mu'}[\alpha + \alpha_0] \rightarrow \mathbf{A}_+^\mu[\alpha]$ and their graded versions $\widehat{\text{inc}}, \widehat{\text{tr}}$.

We still assume $k = 0$ but we do not assume $W_\mu \supset W_{\mu'}$ any more. We define the Zuckerman functors Zuc_0^\pm for this case. Let us identify $\mathbf{A}_+^\mu[\alpha] \simeq \overline{\mathbf{A}}_+^\mu[\beta + \overline{\alpha}]$ and $\mathbf{A}_+^{\mu'}[\alpha + \alpha_k] \simeq \overline{\mathbf{A}}_+^{\mu'}[\beta + \overline{\alpha} + \overline{\alpha}_k + \overline{\alpha}_{k+1}]$. By Lemmas 5.23, 5.27 we have the following inclusions of categories

$$\overline{\mathbf{A}}_+^\mu[\beta + \overline{\alpha}] \subset \overline{A}_+^{\mu^0}[\beta + \overline{\alpha} + \overline{\alpha}_0], \quad \overline{\mathbf{A}}_+^{\mu^0}[\beta + \overline{\alpha} + \overline{\alpha}_0] \supset \overline{\mathbf{A}}_+^{\mu'}[\beta + \overline{\alpha} + \overline{\alpha}_0 + \overline{\alpha}_1].$$

We define the Zuckerman functor $\text{Zuc}_0^+: \mathbf{A}_+^\mu[\alpha] \rightarrow \mathbf{A}_+^{\mu'}[\alpha + \alpha_0]$ as the composition

$$\mathbf{A}_+^\mu[\alpha] \xrightarrow{\text{inc}} \overline{A}_+^{\mu^0}[\beta + \overline{\alpha} + \overline{\alpha}_0] \xrightarrow{b} \overline{\mathbf{A}}_+^{\mu^0}[\beta + \overline{\alpha} + \overline{\alpha}_0] \xrightarrow{\text{tr}} \mathbf{A}_+^{\mu'}[\alpha + \alpha_0].$$

Similarly, we define the Zuckerman functor $\text{Zuc}_0^-: \mathbf{A}_+^{\mu'}[\alpha + \alpha_0] \rightarrow \mathbf{A}_+^\mu[\alpha]$ as the composition

$$\mathbf{A}_+^{\mu'}[\alpha + \alpha_0] \xrightarrow{\text{inc}} \overline{\mathbf{A}}_+^{\mu^0}[\beta + \overline{\alpha} + \overline{\alpha}_0] \xrightarrow{a} \overline{A}_+^{\mu^0}[\beta + \overline{\alpha} + \overline{\alpha}_0] \xrightarrow{\text{tr}} \mathbf{A}_+^\mu[\alpha].$$

Replacing the functors inc , tr , a , b by their graded versions $\widetilde{\text{inc}}$, $\widetilde{\text{tr}}$, \widetilde{a} , \widetilde{b} yields graded versions $\widetilde{\text{Zuc}}_0^+$ and $\widetilde{\text{Zuc}}_0^-$ of the Zuckerman functors.

Now, similarly to Theorem 5.24 we can prove the following.

Theorem 5.28. *Assume that we have $k = 0$ and $W_\mu \supset W_{\mu'}$.*

(a) *The functor $\widetilde{F}_0: D^b(\widetilde{\mathbf{A}}^\nu[\alpha]) \rightarrow D^b(\widetilde{A}^\nu[\alpha + \alpha_0])$ is Koszul dual to the parabolic inclusion functor $\widetilde{\text{inc}}: D^b(\widetilde{\mathbf{A}}_+^\mu[\alpha]) \rightarrow D^b(\widetilde{A}_+^{\mu'}[\alpha + \alpha_0])$.*

(b) *The functor $\widetilde{E}_0: D^b(\widetilde{A}^\nu[\alpha + \alpha_0]) \rightarrow D^b(\widetilde{\mathbf{A}}^\nu[\alpha])$ is Koszul dual to the shifted parabolic truncation functor $\widetilde{\text{tr}}\langle \mu_0 - 1 \rangle: D^b(\widetilde{A}_+^{\mu'}[\alpha + \alpha_0]) \rightarrow D^b(\widetilde{\mathbf{A}}_+^\mu[\alpha])$. \square*

Finally, we get an analogue of Theorem 5.29 in the case $k = 0$.

Theorem 5.29. *Assume that we have $\nu_r > |\alpha|$ for each $r \in [1, l]$ and $e > 2$.*

(a) *The functor $F_0: \mathbf{A}^\nu[\alpha] \rightarrow \mathbf{A}^\nu[\alpha + \alpha_0]$ has a graded lift \widetilde{F}_0 such that the functor $\widetilde{F}_0: D^b(\widetilde{\mathbf{A}}^\nu[\alpha]) \rightarrow D^b(\widetilde{\mathbf{A}}^\nu[\alpha + \alpha_0])$ is Koszul dual to the shifted Zuckerman functor $\widetilde{\text{Zuc}}_0^+\langle \mu_1 \rangle: D^b(\widetilde{\mathbf{A}}_+^\mu[\alpha]) \rightarrow D^b(\widetilde{\mathbf{A}}_+^{\mu'}[\alpha + \alpha_0])$.*

(b) *The functor $E_0: \mathbf{A}^\nu[\alpha + \alpha_0] \rightarrow \mathbf{A}^\nu[\alpha]$ has a graded lift \widetilde{E}_0 such that the functor $\widetilde{E}_0: D^b(\widetilde{\mathbf{A}}^\nu[\alpha + \alpha_0]) \rightarrow D^b(\widetilde{\mathbf{A}}^\nu[\alpha])$ is Koszul dual to the shifted Zuckerman functor $\widetilde{\text{Zuc}}_0^-\langle \mu_0 - 1 \rangle: D^b(\widetilde{\mathbf{A}}_+^\mu[\alpha]) \rightarrow D^b(\widetilde{\mathbf{A}}_+^{\mu'}[\alpha + \alpha_0])$.*

Proof. The proof is similar to the proof of Theorem 5.29. To prove (a) we should consider the diagram as in the proof of Theorem 5.29 with an additional term.

$$\begin{array}{ccccccc}
\overline{\mathbf{A}}^\nu[\beta + \overline{\alpha}] & \xrightarrow{\overline{F}_0} & \overline{\mathbf{A}}^\nu[\beta + \overline{\alpha} + \overline{\alpha}_0] & \longrightarrow & \overline{\mathbf{A}}^\nu[\beta + \overline{\alpha} + \overline{\alpha}_0] & \xrightarrow{\overline{F}_1} & \overline{\mathbf{A}}^\nu[\beta + \overline{\alpha} + \overline{\alpha}_0 + \overline{\alpha}_1] \\
\uparrow & & & & & & \downarrow \\
\mathbf{A}^\nu[\alpha] & \xrightarrow{\quad F_0 \quad} & & & & & \mathbf{A}^\nu[\alpha + \alpha_0]
\end{array}$$

We prove (b) in the same way by considering the diagram obtained from the diagram above by adjointness. Note that in this case we have the adjunction (F_0, E_0) (and not only (E_0, F_0)) because of the assumption on ν . \square

A Generalization of Lemma 3.9

In this appendix we generalize Lemma 3.9. The results of this appendix are not used in the main part of the paper.

A.1 Reduction of the number of idempotents

First, we show that it is possible to reduce the number of idempotents in the quotient in Definition 2.5. Here we use the notation as in Section 2.2.

We fix $\alpha \in Q_I^+$ and put $\bar{\alpha} = \phi(\alpha)$. We say that the sequence $\mathbf{i} \in \bar{T}^{\bar{\alpha}}$ is *almost ordered* if there exists a well-ordered sequence $\mathbf{j} \in \bar{T}^{\bar{\alpha}}$ such that there exist an index r such that $j_r \in \bar{T}_1$ and $\mathbf{i} = s_r(\mathbf{j})$. It is clear from the definition that each almost ordered sequence is unordered because the subsequence (i_1, i_2, \dots, i_r) of \mathbf{i} contains more elements from \bar{T}_2 than from \bar{T}_1 . The following lemma reduces the number of generators of the kernel of $\mathbf{e}R_{\bar{\alpha}, \mathbf{k}}\mathbf{e} \rightarrow S_{\bar{\alpha}, \mathbf{k}}$ (see Definition 2.5).

Lemma A.1. *The kernel of the homomorphism $\mathbf{e}R_{\bar{\alpha}, \mathbf{k}}\mathbf{e} \rightarrow S_{\bar{\alpha}, \mathbf{k}}$ is generated by $e(\mathbf{i})$, where \mathbf{i} runs over the set of all almost ordered sequences in $\bar{T}^{\bar{\alpha}}$.*

Proof. Denote by J the ideal of $\mathbf{e}R_{\bar{\alpha}, \mathbf{k}}\mathbf{e}$ generated by $e(\mathbf{i})$, where \mathbf{i} runs over the set of all almost ordered sequences in $\bar{T}^{\bar{\alpha}}$.

By definition, each element of the kernel of $\mathbf{e}R_{\bar{\alpha}, \mathbf{k}}\mathbf{e} \rightarrow S_{\bar{\alpha}, \mathbf{k}}$ is a linear combination of elements of the form $\mathbf{e}ae(\mathbf{j})\mathbf{b}\mathbf{e}$, where a and b are in $R_{\bar{\alpha}, \mathbf{k}}$ and the sequence \mathbf{j} is unordered. By Remark 2.2, it is enough to prove that for each $\mathbf{i} \in \bar{T}_{\text{ord}}^{\bar{\alpha}}$, $\mathbf{j} \in \bar{T}_{\text{un}}^{\bar{\alpha}}$, $b \in R_{\bar{\alpha}, \mathbf{k}}$ and indices p_1, \dots, p_k the element $e(\mathbf{i})\tau_{p_1} \cdots \tau_{p_k}e(\mathbf{j})\mathbf{b}\mathbf{e}$ is in J . We will prove this statement by induction on k .

Assume that $k = 1$. Write $p = p_1$. The element $e(\mathbf{i})\tau_p e(\mathbf{j})\mathbf{b}\mathbf{e}$ may be nonzero only if $\mathbf{i} = s_p(\mathbf{j})$. This is possible only if the sequence \mathbf{j} is almost ordered. Thus the element $e(\mathbf{i})\tau_p e(\mathbf{j})\mathbf{b}\mathbf{e}$ is in J .

Now, assume that $k > 1$ and that the statement is true for each value $< k$. Set $w = s_{p_1} \cdots s_{p_k}$. We may assume that $\mathbf{i} = w(\mathbf{j})$, otherwise the element $e(\mathbf{i})\tau_{p_1} \cdots \tau_{p_k}e(\mathbf{j})\mathbf{b}\mathbf{e}$ is zero. By assumptions on \mathbf{i} and \mathbf{j} there is an index $r \in [1, d]$ such that $i_r \in \bar{T}_1$ and $w^{-1}(r+1) < w^{-1}(r)$. Thus w has a reduced expression of the form $w = s_r s_{r_1} \cdots s_{r_h}$. This implies that $\tau_{p_1} \cdots \tau_{p_k}e(\mathbf{j})$ is equal to a monomial of the form $\tau_r \tau_{r_1} \cdots \tau_{r_h}e(\mathbf{j})$ modulo monomials of the form $\tau_{q_1} \cdots \tau_{q_t} x_1^{b_1} \cdots x_d^{b_d} e(\mathbf{j})$ with $t < k$, see Remark 2.2. Thus the element $e(\mathbf{i})\tau_{p_1} \cdots \tau_{p_k}e(\mathbf{j})\mathbf{b}\mathbf{e}$ is equal to $e(\mathbf{i})\tau_r \tau_{r_1} \cdots \tau_{r_h}e(\mathbf{j})\mathbf{b}\mathbf{e}$ modulo the elements of the same form $e(\mathbf{i})\tau_{p_1} \cdots \tau_{p_k}e(\mathbf{j})\mathbf{b}\mathbf{e}$ with smaller k . The element $e(\mathbf{i})\tau_r \tau_{r_1} \cdots \tau_{r_h}e(\mathbf{j})\mathbf{b}\mathbf{e}$ is in J because the sequence $s_r(\mathbf{i})$ is almost ordered and the additional terms are in J by the induction assumption. \square

A.2 Categorical representations

In this section we modify slightly the definition of a categorical representation given in Definition 3.7. The only difference is that we use the lattice Q_I instead of X_I . In this section we work with an arbitrary quiver $\Gamma = (I, H)$ without 1-loops.

Let \mathbf{k} be a field. Let \mathcal{C} be a \mathbf{k} -linear Hom-finite category.

Definition A.2. A \mathfrak{g}_I -categorical representation (E, F, x, τ) in \mathcal{C} is the following data:

- (1) a decomposition $\mathcal{C} = \bigoplus_{\alpha \in Q_I} \mathcal{C}_{\alpha}$,
- (2) a pair of biadjoint exact endofunctors (E, F) of \mathcal{C} ,

(3) morphisms of functors $x: F \rightarrow F$, $\tau: F^2 \rightarrow F^2$,

(4) decompositions $E = \bigoplus_{i \in I} E_i$, $F = \bigoplus_{i \in I} F_i$,

satisfying the following conditions.

(a) We have $E_i(\mathcal{C}_\alpha) \subset \mathcal{C}_{\alpha - \alpha_i}$, $F_i(\mathcal{C}_\alpha) \subset \mathcal{C}_{\alpha + \alpha_i}$.

(b) For each $d \in \mathbb{N}$ there is an algebra homomorphism $\psi_d: R_{d, \mathbf{k}} \rightarrow \text{End}(F^d)^{\text{op}}$ such that $\psi_d(e(\mathbf{i}))$ is the projector to $F_{i_d} \cdots F_{i_1}$, where $\mathbf{i} = (i_1, \dots, i_d)$ and

$$\psi_d(x_r) = F^{d-r} x F^{r-1}, \quad \psi_d(\tau_r) = F^{d-r-1} \tau F^{r-1}.$$

(c) For each $M \in \mathcal{C}$ the endomorphism of $F(M)$ induced by x is nilpotent.

Now, fix a decomposition $I = I_0 \amalg I_1$ as in Section 2.2. We consider the quiver $\bar{\Gamma} = (\bar{I}, \bar{H})$ and the map ϕ as in Section 2.2. To distinguish the elements of Q_I and $Q_{\bar{I}}$, we write $Q_{\bar{I}} = \bigoplus_{i \in \bar{I}} \mathbb{Z} \bar{\alpha}_i$. For each $\alpha \in Q_I$ we set $\bar{\alpha} = \phi(\alpha) \in Q_{\bar{I}}$. (See Section 2.2 for the notation.) However we can sometimes use the symbol $\bar{\alpha}$ for an arbitrary element of $Q_{\bar{I}}$ that is not associated with some α in Q_I . Let $\bar{\mathcal{C}}$ be a Hom-finite abelian \mathbf{k} -linear category. Let $\bar{E} = \bigoplus_{i \in \bar{I}} \bar{E}_i$ and $\bar{F} = \bigoplus_{i \in \bar{I}} \bar{F}_i$ be endofunctors defining a categorical $\mathfrak{g}_{\bar{I}}$ -representation in $\bar{\mathcal{C}}$. Let $\bar{\psi}_d: R_{d, \mathbf{k}}(\bar{\Gamma}) \rightarrow \text{End}(\bar{F}^d)^{\text{op}}$ be the corresponding algebra homomorphism. We set $\bar{F}_{\mathbf{i}} = \bar{F}_{i_d} \cdots \bar{F}_{i_1}$ for any tuple $\mathbf{i} = (i_1, \dots, i_d) \in \bar{I}^d$ and $\bar{F}_{\bar{\alpha}} = \bigoplus_{\mathbf{i} \in \bar{I}^{\bar{\alpha}}} \bar{F}_{\mathbf{i}}$ for any element $\bar{\alpha} \in Q_{\bar{I}}^+$. If $|\bar{\alpha}| = d$, let $\bar{\psi}_{\bar{\alpha}}: R_{\bar{\alpha}, \mathbf{k}} \rightarrow \text{End}(\bar{F}_{\bar{\alpha}})^{\text{op}}$ be the $\bar{\alpha}$ -component of $\bar{\psi}_d$.

Assume that \mathcal{C} is an abelian subcategory of $\bar{\mathcal{C}}$ satisfying the following conditions:

(a) \mathcal{C} is stable by \bar{F}_i, \bar{E}_i for each $i \in I_0$,

(b) \mathcal{C} is stable by $\bar{F}_{i^2} \bar{F}_{i^1}, \bar{E}_{i^1} \bar{E}_{i^2}$ for each $i \in I_1$,

(c) we have $\bar{F}_{i^2}(\mathcal{C}) = 0$ for each $i \in I_1$,

(d) we have $\mathcal{C} = \bigoplus_{\alpha \in Q_I} \mathcal{C} \cap \bar{\mathcal{C}}_{\bar{\alpha}}$.

By (d), we get a decomposition $\mathcal{C} = \bigoplus_{\alpha \in Q_I} \mathcal{C}_\alpha$, where $\mathcal{C}_\alpha = \mathcal{C} \cap \bar{\mathcal{C}}_{\bar{\alpha}}$. For each $i \in I$ we consider the following endofunctors \bar{E}_i, F_i of \mathcal{C} :

$$F_i = \begin{cases} \bar{F}_i|_{\mathcal{C}} & \text{if } i \in I_0, \\ \bar{F}_{i^2} \bar{F}_{i^1}|_{\mathcal{C}} & \text{if } i \in I_1, \end{cases}$$

$$E_i = \begin{cases} \bar{E}_i|_{\mathcal{C}} & \text{if } i \in I_0, \\ \bar{E}_{i^1} \bar{E}_{i^2}|_{\mathcal{C}} & \text{if } i \in I_1. \end{cases}$$

Similarly to the notations above we set $F_{\mathbf{i}} = F_{i_d} \cdots F_{i_1}$ for any tuple $\mathbf{i} = (i_1, \dots, i_d) \in I^d$ and $F_\alpha = \bigoplus_{\mathbf{i} \in I^\alpha} F_{\mathbf{i}}$ for any element $\alpha \in Q_I^+$. Note that we have $F_{\mathbf{i}} = \bar{F}_{\phi(\mathbf{i})}|_{\mathcal{C}}$ for each $\mathbf{i} \in I^\alpha$.

Let $\alpha \in \mathbb{N}I$. We have

$$F_\alpha = \bigoplus_{\mathbf{i} \in \overline{I}_{\text{ord}}^\alpha} \overline{F}_{\mathbf{i}}|_{\mathcal{C}}.$$

The homomorphism $\overline{\psi}_\alpha$ yields a homomorphism $\mathbf{e}R_{\overline{\alpha}, \mathbf{k}}\mathbf{e} \rightarrow \text{End}(F_\alpha)^{\text{op}}$, where $\mathbf{e} = \sum_{\mathbf{i} \in \overline{I}_{\text{ord}}^\alpha} e(\mathbf{i})$.

Since the category \mathcal{C} satisfies (a), (b) and (c), for each almost ordered sequence $\mathbf{i} = (i_1, \dots, i_d) \in I^\alpha$ we have $\overline{F}_{i_d} \cdots \overline{F}_{i_1}(\mathcal{C}) = 0$. By Lemma A.1, this implies that the homomorphism $\mathbf{e}R_{\overline{\alpha}, \mathbf{k}}\mathbf{e} \rightarrow \text{End}(F_\alpha)^{\text{op}}$ factors through a homomorphism $S_{\overline{\alpha}, \mathbf{k}} \rightarrow \text{End}(F_\alpha)^{\text{op}}$. Let us call it $\overline{\psi}'_\alpha$. Then we can define an algebra homomorphism $\psi_\alpha: R_{\alpha, \mathbf{k}} \rightarrow \text{End}(F_\alpha)^{\text{op}}$ by setting $\psi_\alpha = \overline{\psi}'_\alpha \circ \Phi_{\alpha, \mathbf{k}}$.

Now, Theorem 2.11 implies the following result.

Lemma A.3. *For each abelian subcategory $\mathcal{C} \subset \overline{\mathcal{C}}$ as above, that satisfies (a) – (d), we have a categorical representation of \mathfrak{g}_I in \mathcal{C} given by functors F_i , E_i and the algebra homomorphisms $\psi_\alpha: R_{\alpha, \mathbf{k}} \rightarrow \text{End}(F_\alpha)^{\text{op}}$. \square*

Remark A.4. Assume that the category $\overline{\mathcal{C}}$ is such that we have $\overline{\mathcal{C}}_\alpha = 0$ whenever $\alpha = \sum_{i \in \overline{I}} d_i \alpha_i \in Q_{\overline{I}}$ is such that $d_{i_1} < d_{i_2}$ for some $i \in I_1$. In this case the subcategory $\mathcal{C} \subset \overline{\mathcal{C}}$ defined by $\mathcal{C} = \bigoplus_{\alpha \in Q_I} \overline{\mathcal{C}}_\alpha$ satisfies (a) – (d).

B The geometric construction of the isomorphism Φ

The goal of this appendix is to give a geometric construction of the isomorphism Φ in Theorem 2.11.

B.1 The geometric construction of the KLR algebra

Let \mathbf{k} be a field. Let $\Gamma = (I, H)$ be a quiver without 1-loops. See Section 2.1 for the notations related with quivers. For an arrow $h \in H$ we will write h' and h'' for its source and target respectively. Fix $\alpha = \sum_{i \in I} d_i \alpha_i \in Q_I^+$ and set $d = |\alpha|$. Set also

$$E_\alpha = \bigoplus_{h \in H} \text{Hom}(V_{h'}, V_{h''}), \quad V_i = \mathbb{C}^{d_i}, \quad V = \bigoplus_{i \in I} V_i.$$

The group $G_\alpha = \prod_{i \in I} GL(V_i)$ acts on E_α by base changes.
Set

$$I^\alpha = \{\mathbf{i} = (i_1, \dots, i_d) \in I^d; \sum_{r=1}^d \alpha_{i_r} = \alpha\}.$$

We denote by $F_{\mathbf{i}}$ the variety of all flags

$$\phi = (\{0\} = V^0 \supset V^1 \supset \dots \supset V^d = V)$$

in V that are homogeneous with respect to the decomposition $V = \bigoplus_{i \in I} V_i$ and such that the I -graded vector space V^{r-1}/V^r has graded dimension i_r for $r \in [1, d]$. We denote by \tilde{F}_i the variety of pairs $(x, \phi) \in E_\alpha \times F_i$ such that x preserves ϕ , i.e., we have $x(V^r) \subset V^r$ for $r \in \{0, 1, \dots, m\}$. Let π_i be the natural projection from \tilde{F}_i to E_α , i.e., $\pi_i : \tilde{F}_i \rightarrow E_\alpha$, $(x, \phi) \mapsto x$. For $\mathbf{i}, \mathbf{j} \in I^\alpha$ we denote by $Z_{\mathbf{i}, \mathbf{j}}$ the variety of triples $(x, \phi_1, \phi_2) \in E_\alpha \times F_i \times F_j$ such that x preserves ϕ_1 and ϕ_2 (i.e., we have $Z_{\mathbf{i}, \mathbf{j}} = \tilde{F}_i \times_{E_\alpha} \tilde{F}_j$). Set

$$Z_\alpha = \coprod_{\mathbf{i}, \mathbf{j} \in I^\alpha} Z_{\mathbf{i}, \mathbf{j}}, \quad \tilde{F}_\alpha = \coprod_{\mathbf{i} \in I^\alpha} \tilde{F}_i.$$

We have an algebra structure on $H_*^{G_\alpha}(Z_\alpha, \mathbf{k})$ with respect to the convolution product with respect to the inclusion $Z_\alpha \subset \tilde{F}_\alpha \times \tilde{F}_\alpha$. Here $H_*^{G_\alpha}(\bullet, \mathbf{k})$ denotes the G_α -equivariant Borel-Moore homology with coefficients in \mathbf{k} . See [5, Sec. 2.7] for the definition of the convolution product.

The following result is proved by Rouquier [17] and by Varagnolo-Vasserot [23] in the situation $\text{char } \mathbf{k} = 0$. See [12] for the proof over an arbitrary field.

Proposition B.1. *There is an algebra isomorphism $R_{\alpha, \mathbf{k}}(\Gamma) \simeq H_*^{G_\alpha}(Z_\alpha, \mathbf{k})$. Moreover, for each $\mathbf{i}, \mathbf{j} \in I^\alpha$, the vector subspace $e(\mathbf{i})R_{\alpha, \mathbf{k}}(\Gamma)e(\mathbf{j}) \subset R_{\alpha, \mathbf{k}}(\Gamma)$ corresponds to the vector subspace $H_*^{G_\alpha}(Z_{\mathbf{i}, \mathbf{j}}, \mathbf{k}) \subset H_*^{G_\alpha}(Z_\alpha, \mathbf{k})$.*

□

B.2 The geometric construction of the isomorphism Φ

Fix a decomposition $I = I_0 \coprod I_1$ as in Section 2.2 and consider the quiver $\bar{\Gamma} = (\bar{I}, \bar{H})$ as in Section 2.2. Fix $\alpha \in Q_I^+$ and consider $\bar{\alpha} = \phi(\alpha) \in Q_{\bar{I}}^+$ as in Section 2.2.

We start from the variety $Z_{\bar{\alpha}}$ defined with respect to the quiver $\bar{\Gamma}$. By Proposition B.1, we have an algebra isomorphism $R_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma}) \simeq H_*^{G_{\bar{\alpha}}}(Z_{\bar{\alpha}}, \mathbf{k})$. We have an obvious projection $p : Z_{\bar{\alpha}} \rightarrow E_{\bar{\alpha}}$ defined by $(x, \phi_1, \phi_2) \mapsto x$. For each $i \in I_1$ denote by h_i the unique arrow in $\bar{\Gamma}$ that goes from i^1 to i^2 . Consider the following open subset of $E_{\bar{\alpha}}$: $E_{\bar{\alpha}}^0 = \{x \in E_{\bar{\alpha}}; x_{h_i} \text{ is invertible } \forall i \in I_1\}$. Set $Z_{\bar{\alpha}}^0 = p^{-1}(E_{\bar{\alpha}}^0)$. The pullback with respect to the inclusion $Z_{\bar{\alpha}}^0 \subset Z_{\bar{\alpha}}$ yields an algebra homomorphism $H_*^{G_{\bar{\alpha}}}(Z_{\bar{\alpha}}, \mathbf{k}) \rightarrow H_*^{G_{\bar{\alpha}}}(Z_{\bar{\alpha}}^0, \mathbf{k})$ (see [5, Lem. 2.7.46]).

Remark B.2. If the sequence $\mathbf{i} \in \bar{I}^{\bar{\alpha}}$ is unordered, then a flag from F_i is never preserved by an element from $E_{\bar{\alpha}}^0$. This implies that $Z_{\mathbf{i}, \mathbf{j}} \cap Z_{\bar{\alpha}}^0 = \emptyset$ if \mathbf{i} or \mathbf{j} is unordered. Thus for each $\mathbf{i} \in \bar{I}_{\text{un}}^{\bar{\alpha}}$, the idempotent $e(\mathbf{i})$ is in the kernel of the homomorphism $H_*^{G_{\bar{\alpha}}}(Z_{\bar{\alpha}}, \mathbf{k}) \rightarrow H_*^{G_{\bar{\alpha}}}(Z_{\bar{\alpha}}^0, \mathbf{k})$.

Let \mathbf{e} be the idempotent as in Definition 2.5. Consider the following subset of $Z_{\bar{\alpha}}$:

$$Z'_{\bar{\alpha}} = \coprod_{\mathbf{i}, \mathbf{j} \in \bar{I}_{\text{ord}}^{\bar{\alpha}}} Z_{\mathbf{i}, \mathbf{j}}.$$

The algebra isomorphism $R_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma}) \simeq H_*^{G^\pi}(Z_{\bar{\alpha}}, \mathbf{k})$ above restricts an algebra isomorphism $\mathbf{e}R_{\bar{\alpha}}(\bar{\Gamma})\mathbf{e} \simeq H_*^{G^\pi}(Z'_{\bar{\alpha}}, \mathbf{k})$.

Now, set $Z_\alpha^0 = Z'_\alpha \cap Z_\alpha^0$. Similarly to the construction above, we have an algebra homomorphism $H_*^{G^\pi}(Z'_{\bar{\alpha}}, \mathbf{k}) \rightarrow H_*^{G^\pi}(Z_\alpha^0, \mathbf{k})$. By Remark B.2, the kernel of this homomorphism contains the kernel of $\mathbf{e}R_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma})\mathbf{e} \rightarrow R_{\alpha, \mathbf{k}}(\Gamma)$ (see Theorem 2.11). The following result implies that these kernels are the same.

Lemma B.3. *We have the following algebra isomorphism $R_{\alpha, \mathbf{k}}(\Gamma) \simeq H_*^{G^\pi}(Z_\alpha^0, \mathbf{k})$.*

Proof. For each $i \in I_0$ we identify $V_i \simeq V_{i_0}$. For each $i \in I_1$ we identify $V_i \simeq V_{i_1} \simeq V_{i_2}$. We have a diagonal inclusion $G_\alpha \subset G_{\bar{\alpha}}$, i.e., the component $GL(V_i)$ of G_α with $i \in I_0$ goes to $GL(V_{i_0})$ and the component $GL(V_i)$ with $i \in I_1$ goes diagonally to $GL(V_{i_1}) \times GL(V_{i_2})$.

Set $G_\alpha^{\text{bis}} = \prod_{i \in I_1} GL(V_{i_2}) \subset G_{\bar{\alpha}}$. We have an obvious group isomorphism $G_{\bar{\alpha}} \simeq G_\alpha \times G_\alpha^{\text{bis}}$. (Note that this isomorphism does not identify G_α with the diagonal subgroup of G_α . It sends the component $GL(V_i)$ of G_α with $i \in I_1$ to $GL(V_{i_1})$.)

Let us denote by X the choice of isomorphisms $V_{i_1} \simeq V_{i_2}$ mentioned above. Let $E_{\bar{\alpha}}^X$ be the subset of $E_{\bar{\alpha}}$ that contains only $x \in E_{\bar{\alpha}}$ such that for each $i \in I_1$ the component x_{h_i} is the isomorphism chosen in X .

The group G_α^{bis} acts freely on $E_{\bar{\alpha}}^0$ such that each orbit intersects $E_{\bar{\alpha}}^X$ once. This implies that we have an isomorphism of algebraic varieties $G_\alpha^{\text{bis}} \times E_{\bar{\alpha}}^X \simeq E_{\bar{\alpha}}^0$ given by $(g, x) \mapsto gx$. Now, set $Z_{\bar{\alpha}}^X = p^{-1}(E_{\bar{\alpha}}^X)$. As above, we have $G_{\bar{\alpha}}^{\text{bis}} \times Z_{\bar{\alpha}}^X \simeq Z_{\bar{\alpha}}^0$. Note that the group G_α viewed as a diagonal subgroup of $G_{\bar{\alpha}}$ preserves $Z_{\bar{\alpha}}^X$. We get the following chain of algebra isomorphisms

$$H_*^{G^\pi}(Z'_{\bar{\alpha}}, \mathbf{k}) \simeq H_*^{G_\alpha \times G_\alpha^{\text{bis}}}(G_\alpha^{\text{bis}} \times Z_{\bar{\alpha}}^X, \mathbf{k}) \simeq H_*^{G_\alpha}(Z_{\bar{\alpha}}^X, \mathbf{k}).$$

To complete the proof we have to show that the G_α -variety $Z_{\bar{\alpha}}^X$ is isomorphic to Z_α . Each element of $I_{\text{ord}}^{\bar{\alpha}}$ is of the form $\phi(\mathbf{i})$ for a unique $\mathbf{i} \in I^\alpha$, where ϕ is as in Section 2.2. Let us abbreviate $\mathbf{i}' = \phi(\mathbf{i})$. By definition we have

$$Z'_{\bar{\alpha}} = \coprod_{\mathbf{i}, \mathbf{j} \in I^\alpha} Z_{\mathbf{i}', \mathbf{j}'},$$

Set $Z_{\mathbf{i}', \mathbf{j}'}^X = Z_{\mathbf{i}', \mathbf{j}'} \cap Z_{\bar{\alpha}}^X$. We have an obvious isomorphism of G_α -varieties $Z_{\mathbf{i}', \mathbf{j}'}^X \simeq Z_{\mathbf{i}, \mathbf{j}}$. (Beware, the variety $Z_{\mathbf{i}, \mathbf{j}}$ is defined with respect to the quiver Γ and the variety $Z_{\mathbf{i}', \mathbf{j}'}$ is defined with respect to the quiver $\bar{\Gamma}$.) Taking the union by all $\mathbf{i}, \mathbf{j} \in I^\alpha$ yields an isomorphism of G_α -varieties $Z_{\bar{\alpha}}^X \simeq Z_\alpha$. \square

Corollary B.4. *We have the following commutative diagram.*

$$\begin{array}{ccc} \mathbf{e}R_{\bar{\alpha}, \mathbf{k}}(\bar{\Gamma})\mathbf{e} & \longrightarrow & R_{\alpha, \mathbf{k}}(\Gamma) \\ \downarrow & & \downarrow \\ H_*^{G^\pi}(Z'_{\bar{\alpha}}, \mathbf{k}) & \longrightarrow & H_*^{G^\pi}(Z_\alpha^0, \mathbf{k}). \end{array}$$

Here the left vertical map is the isomorphism from Proposition B.1, the right vertical map is the isomorphism from Lemma B.3, the top horizontal map is obtained from Theorem 2.11 and the bottom horizontal map is the pullback with respect to the inclusion $Z_{\alpha}^{\prime 0} \subset Z_{\alpha}^{\prime}$.

Proof. The result follows directly from Lemma B.3. The commutativity of the diagram is easy to see on the generators of $R_{\alpha, \mathbf{k}}(\overline{\Gamma})$. \square

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